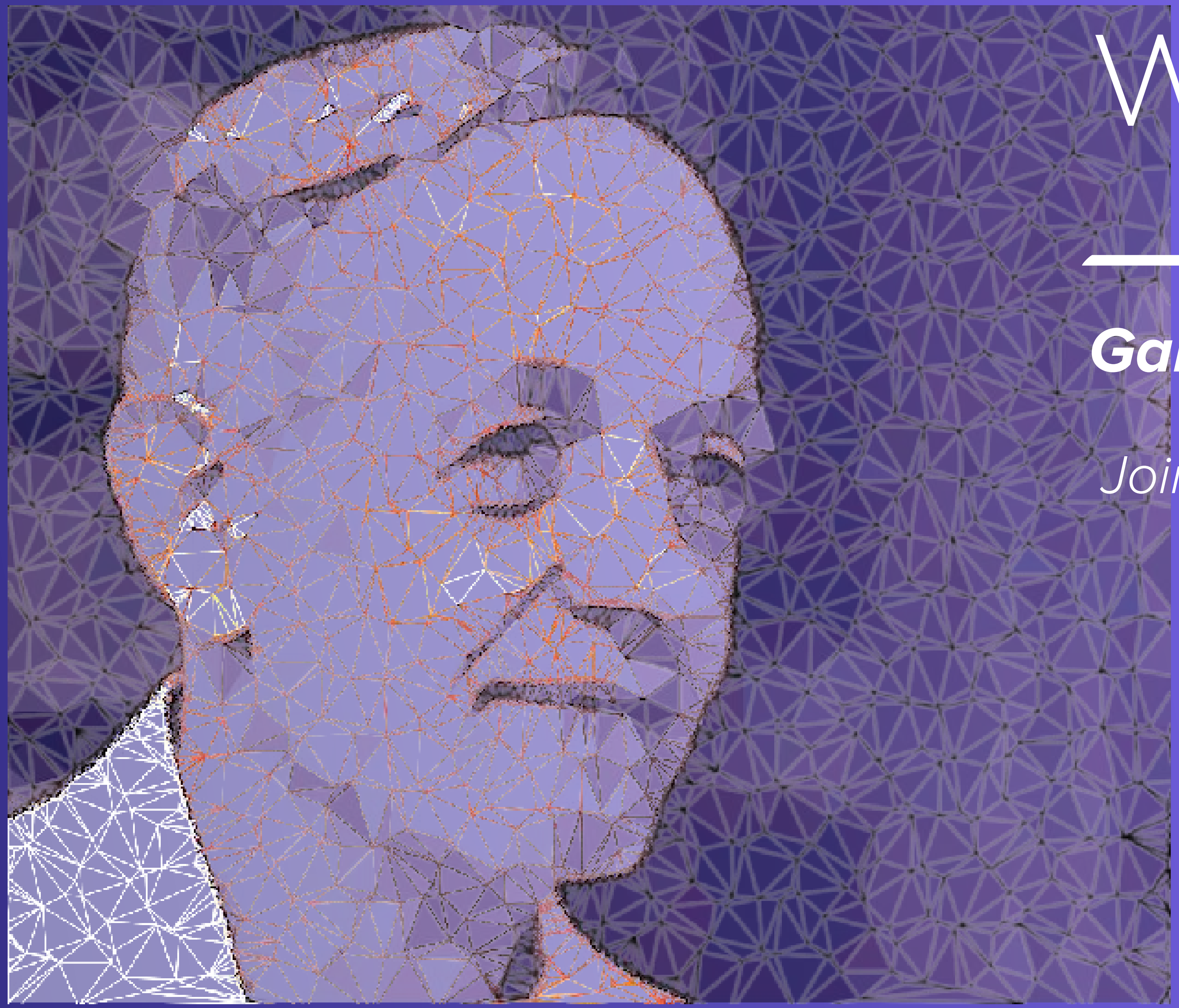


When Nash meets Stackelberg

Gabriele Dragotto

Joint work with M. Carvalho, F. Feijoo, A. Lodi, S. Sankaranarayanan

Consider a *Bagel Shop*

St Viateur Bagel

sells their Bagels in a market in order to profit

Simultaneous
Nash Game

And *competes* with *Fairmount Bagel*

Hence, they play a simultaneous game with Bagels

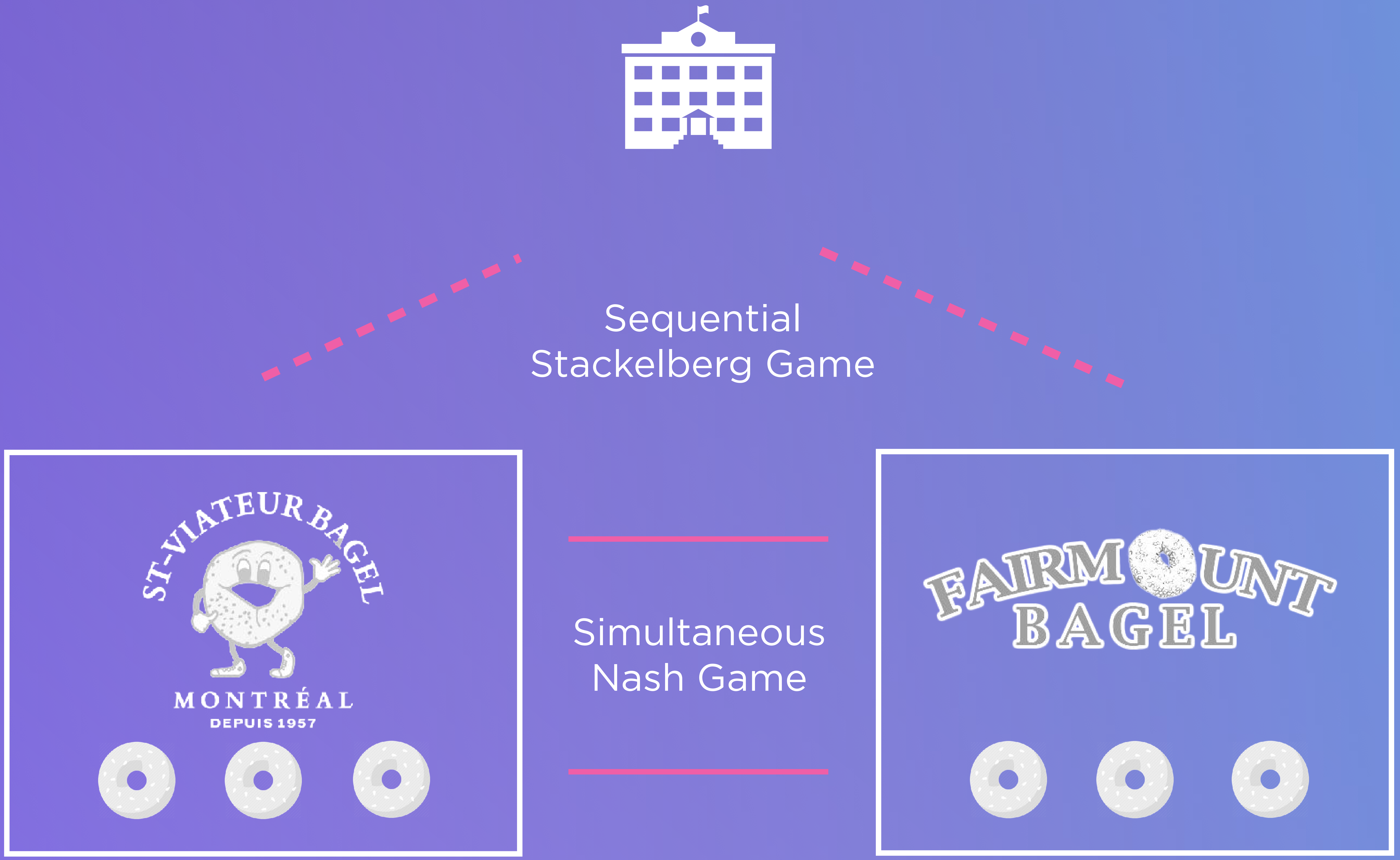
Montréal taxes their bagels

Since ovens are *polluting* the city air

ST-VIATEUR BAGEL
MONTRÉAL
DEPUIS 1957

Simultaneous
Nash Game

FAIRMOUNT BAGEL



Montréal regulates the market

And playing a sequential game with the Bagel Shops

Montréal competes with New York

The cities play another simultaneous game among themselves

We call this *Nash Game Among Stackelberg Leaders (NASP)*

What if....

Bagel shop are instead energy producers

And cities are regulating governments

Background

Stackelberg Game

(Stackelberg, 1934;
Candler and Norto, 1977)

In many cases, at least $\mathscr{N}\mathscr{P}$-hard

The feasible set for the leader is often *non-convex,* and *non-connected*

💡 *Basu et al. (2020)* prove that under certain assumption, $\mathscr{F}$ is a union of polyhedra

Background

Nash Equilibrium

(Nash, 1950, 1951)

Each player solves an optimization problem P^i depending on its decisions x^i and the one of other players x^{-i}

When each agent plays a *mutually optimal strategy* x^i wrt the strategies x^{-i} of all the other players, then we have a so-called *Nash Equilibrium*

Then, $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^n)$ is a *Mixed Nash Equilibrium (MNE)* iff

No player has an *incentive to deviate* from its equilibrium strategy $\bar{x}^i$ *given the other player choices* $\bar{x}^{-i}$

MNE: "play" *a single* strategy with probability 1

PNE: "play" multiple strateg*ies* where probabilities sum up to 1

Some Results


Complexity
NASPs
Convexification of Games
Algorithms

Complexity

There are *three* fundamental complexity theorems for *NASPs*

Σ_2^p *Hardness*

Theorem *(Carvalho, D., Feijoo, Lodi, Sankaranarayanan, 2019)*

Given a NASP with 2 leaders and 1 follower each, the followers solve a linear program and the leaders have linear objectives:

- It is Σ_2^p-hard to decide if the game has an *MNE*
- It is Σ_2^p-hard to decide if the game has an *PNE* even if all leaders' feasible regions are bounded
- If each player feasible region is *bounded*, then there exists an *MNE*

The problem is not PLS-complete as many other equilibrium problems!

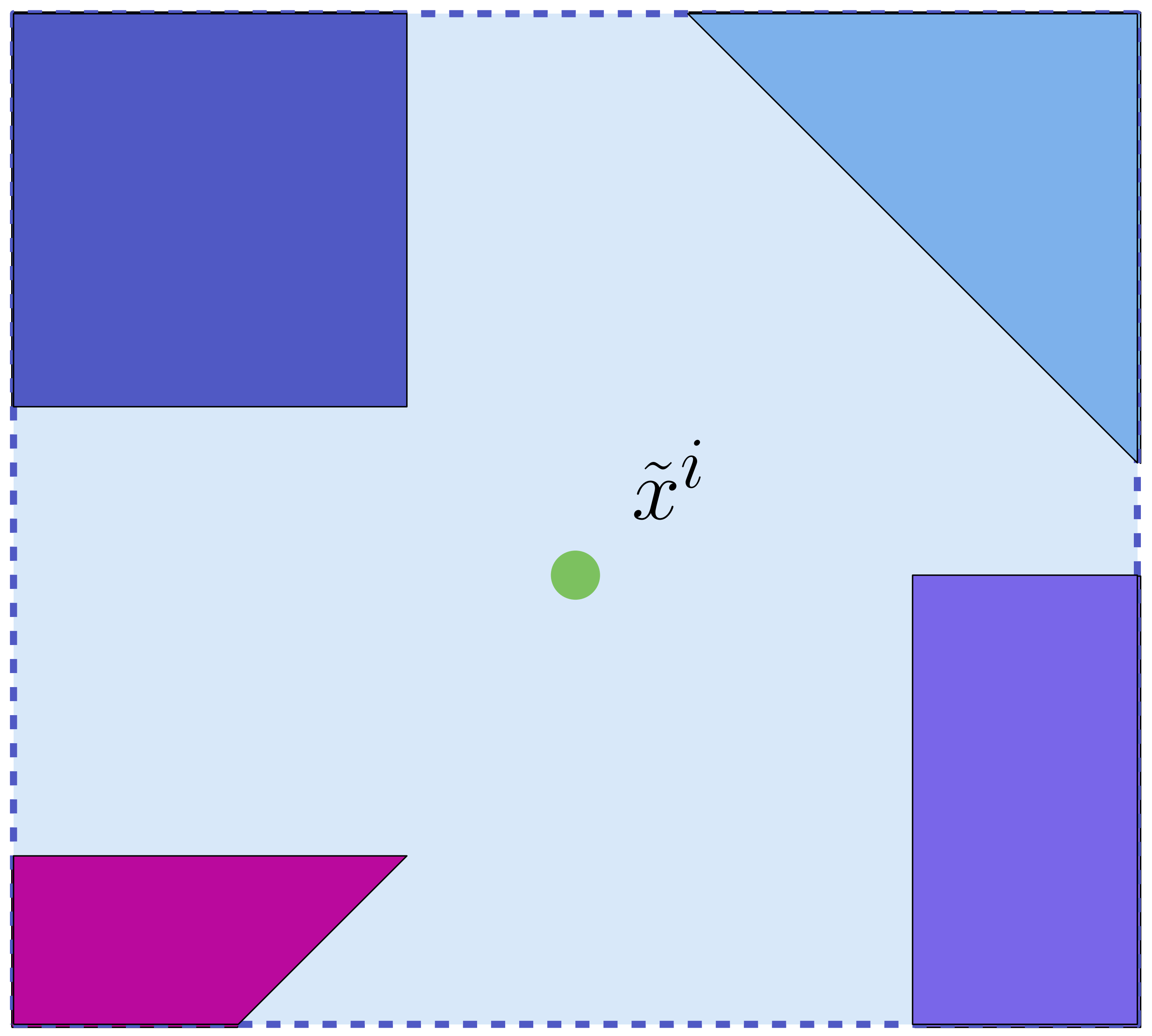

Complexity
NASPs
Convexification of Games
Algorithms

Convexification of Games

Every player i has a non-convex feasible region $\mathscr{F}_i$, made of a *union of polyhedra*

We can use Balas's to retrieve their *convex-hull* $\mathrm{cl}\ \mathrm{conv}(\mathscr{F}_i)$

We solve it and find a solution $\tilde{x}^i$ in the convex-hull,
but not within any of the *original polyhedra.*



Convexification of Games

We solve it and find a solution $\tilde{x}^i$ in the convex-hull, but not within any of the *original polyhedra.*

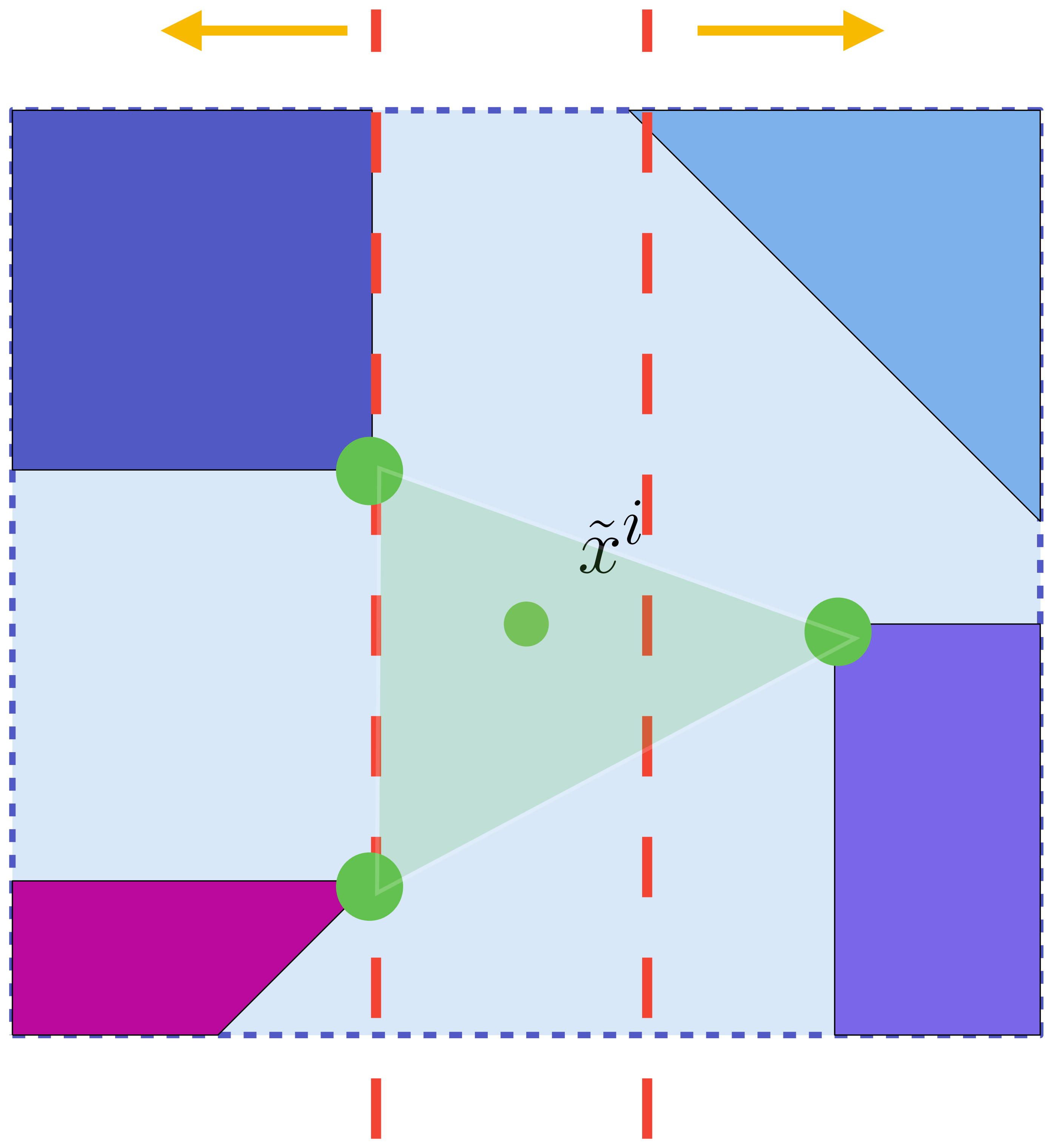
If it was a MIP

The solution is not feasible. We would search for a disjunction and *cut it off!*

MNE interpretation

Each point in cl conv$(\mathscr{F}_i) \setminus \mathscr{F}_i$ can be expressed as a convex combination of points *strictly laying in* $\mathscr{F}_i$

If points in $\mathscr{F}_i$ are *pure strategies,* then cl conv$(\mathscr{F}_i) \setminus \mathscr{F}_i$ contains *mixed strategies!*




Complexity
NASPs
Convexification of Games
Algorithms

Algorithms

All the following algorithms are valid for *NASPs as well as for* generic Stackelberg Games!

Algorithms

A full enumeration

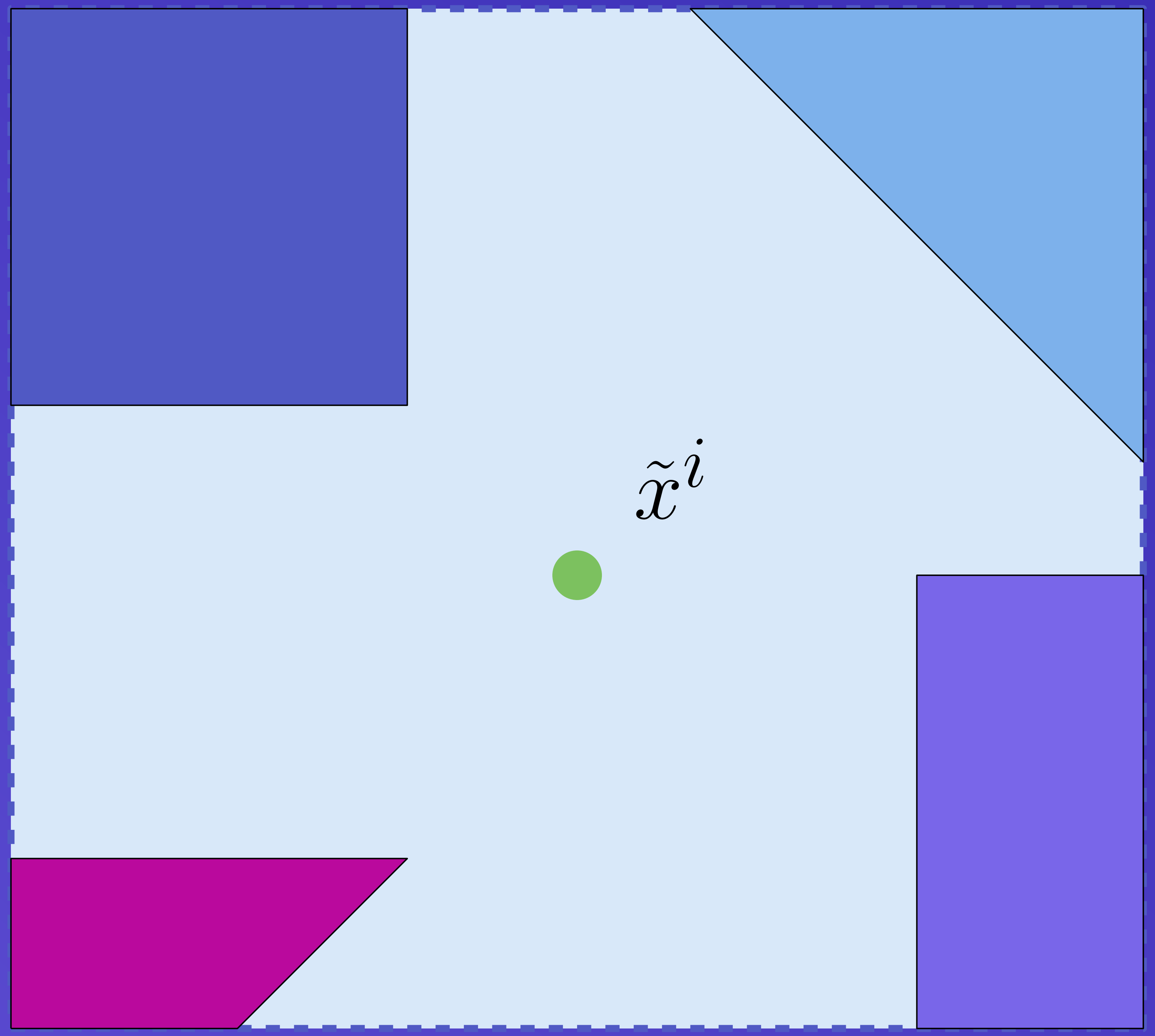
For each player i, go for a full enumeration of its feasible region $\mathcal{F}_i$

- Find $\mathcal{F}_i^* = \text{cl conv}(\mathcal{F}_i)$ using Balas'

Solve the *Nash Game* on $\mathcal{F}^* = \mathcal{F}_1^* \times \ldots \times \mathcal{F}_n^*$ for $\tilde{x} = (\tilde{x}^1, \ldots, \tilde{x}^n)$

Theorem *(Carvalho, D., Feijoo, Lodi, Sankaranarayanan, 2019)*

The full enumeration algorithm terminates finitely either with a MNE or a certificate of non-existence.



Algorithms

Inner Approximation

Instead of enumerating all the polyhedra, start with one polyhedron k for each player i. Try to find an Equilibrium

Poly ^i_k

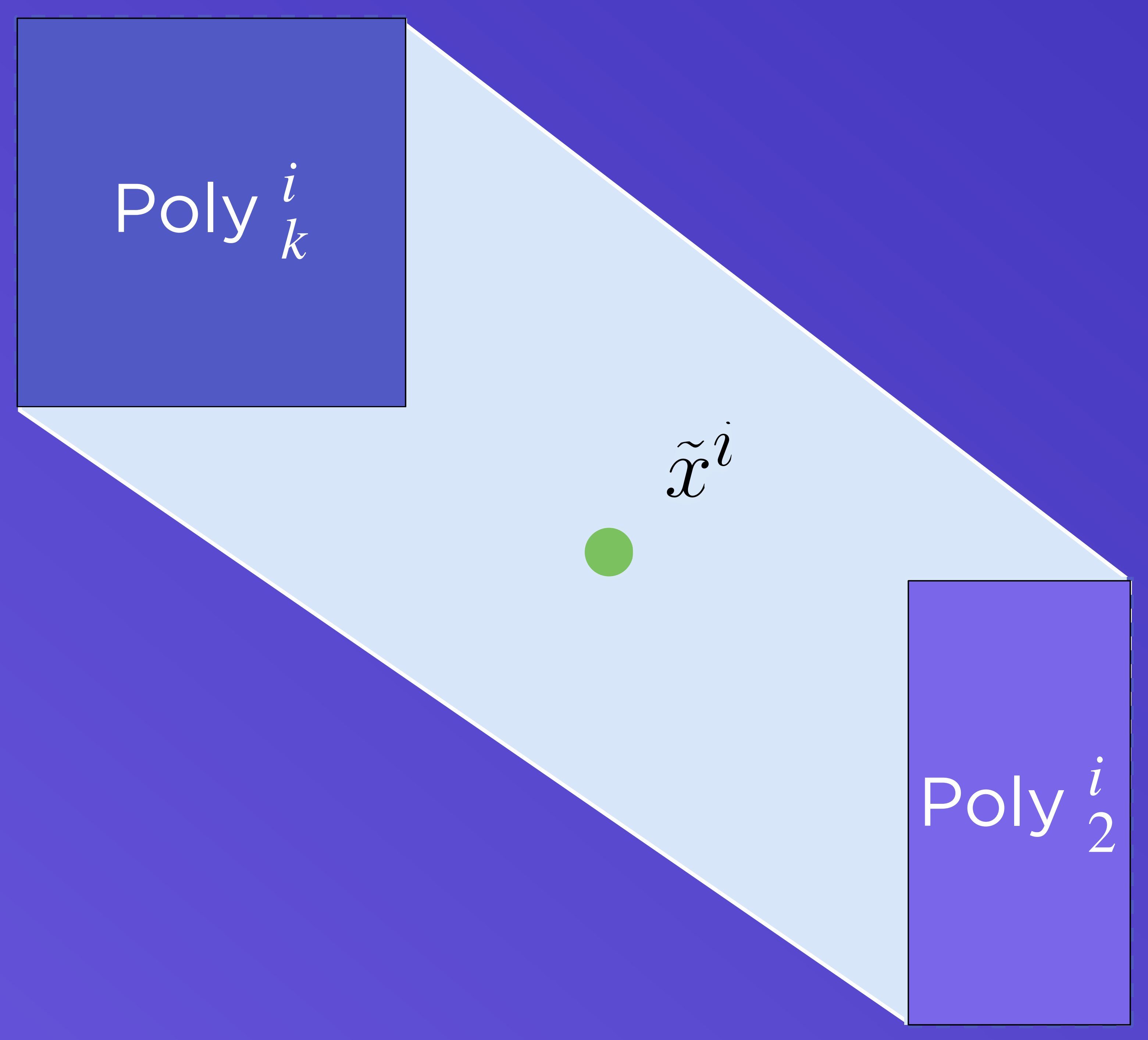
There exists an *MNE* $\tilde{x}$*:*
check if any player can deviate. This can be done with a simple *MIP* by fixing $\tilde{x}^{-i}$ and solving for x^i:

- If no deviation exists, then TERMINATE. We found an *MNE*
- Otherwise, *add the polyhedra containing the deviations* to the approximation and repeat.

Algorithms

Inner Approximation

- Otherwise, *add the polyhedra containing the deviations* to the approximation and repeat



Algorithms

Inner Approximation

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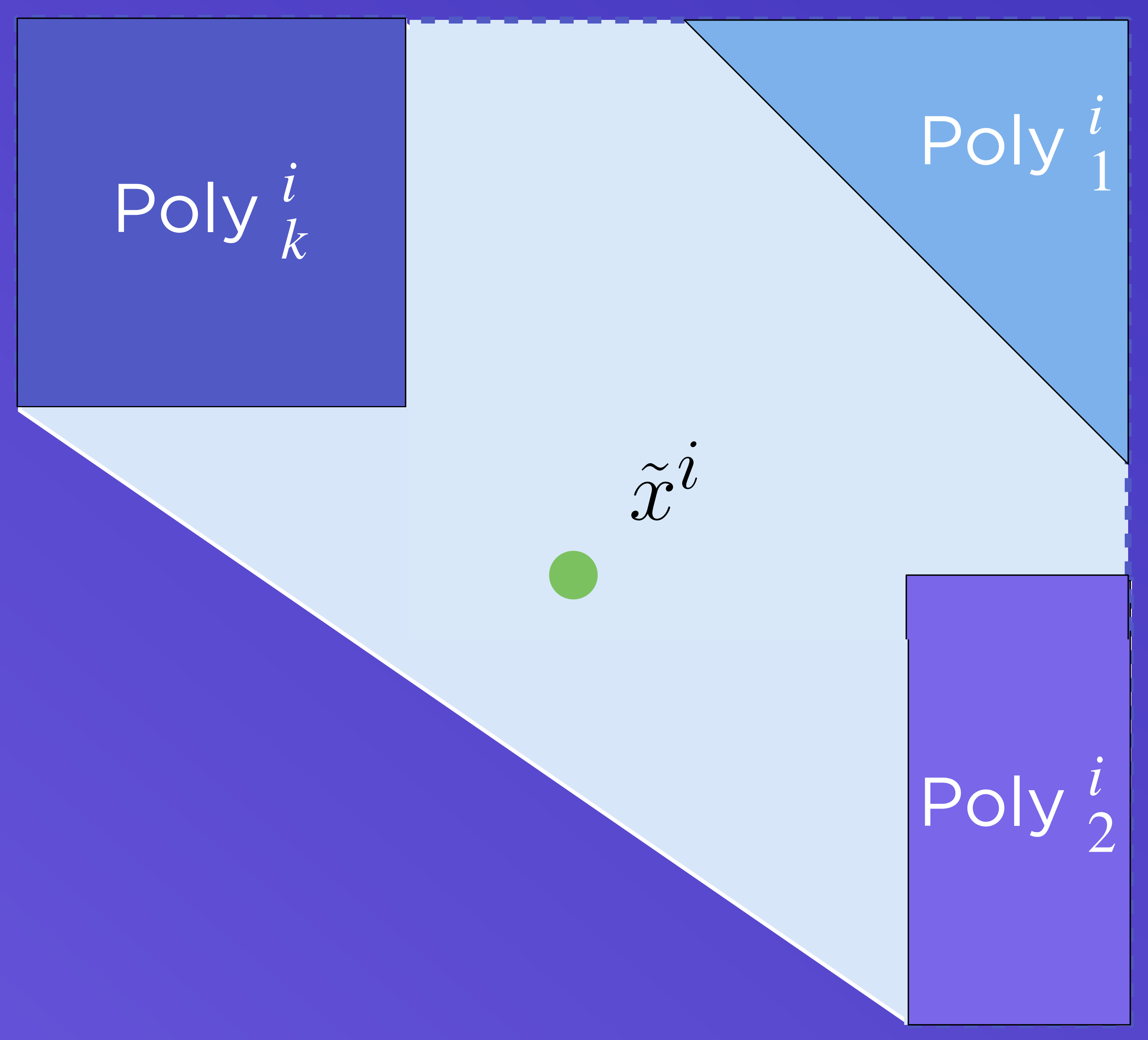

Algorithms

Inner Approximation

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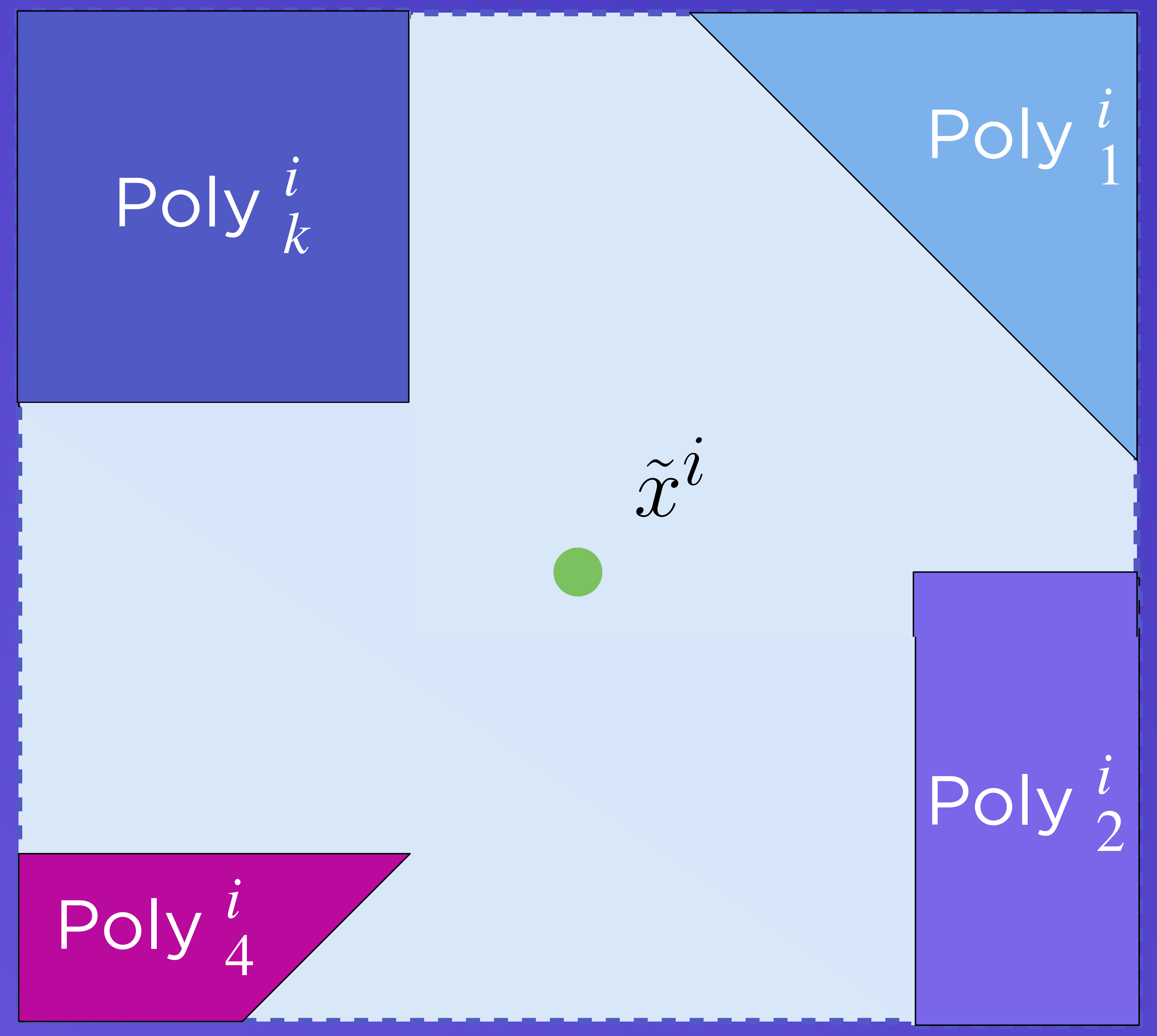


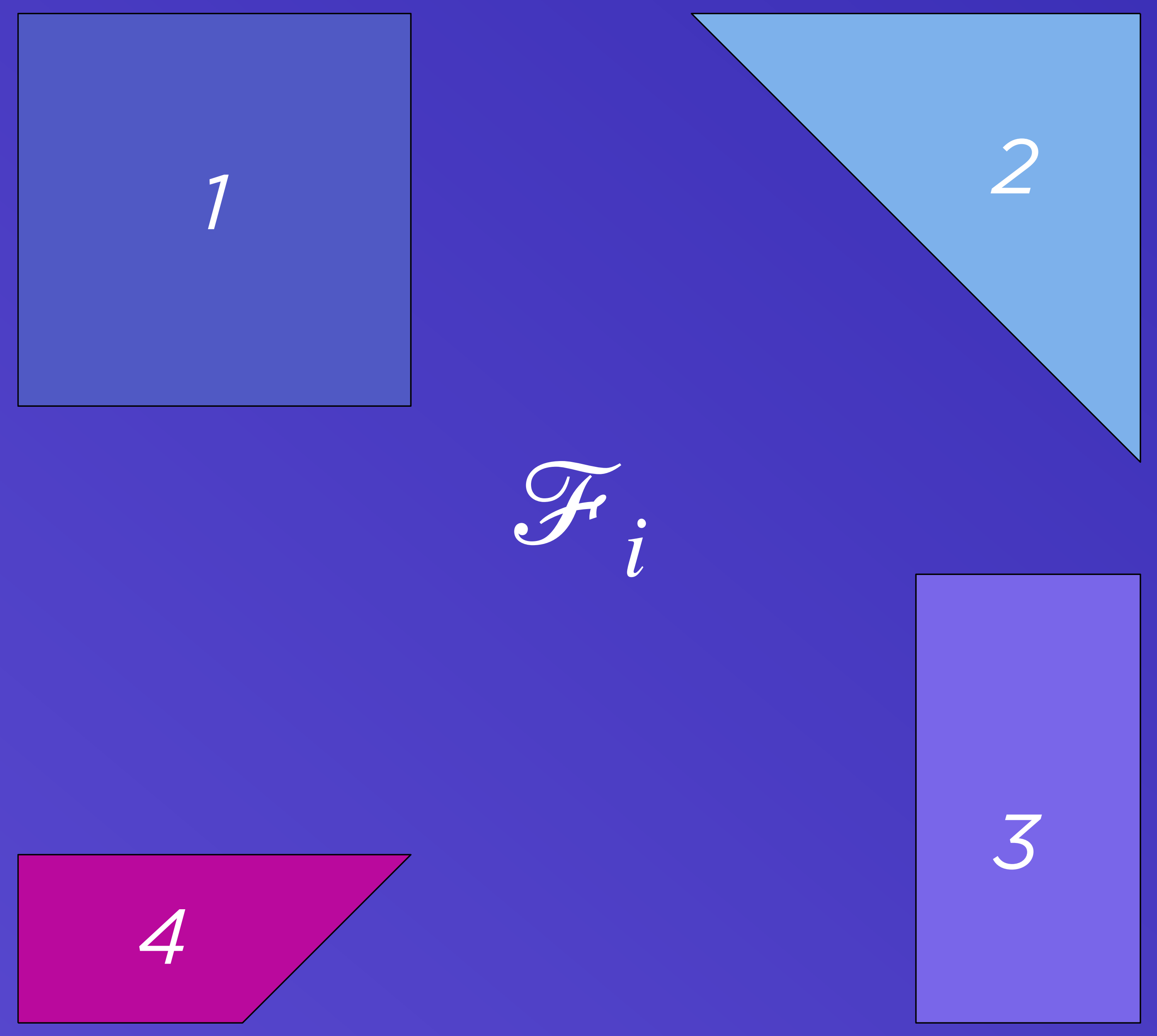
Algorithms

Combinatorial Heuristic

If we are interested only in *PNEs,* then the equilibrium $\tilde{x}$ strictly lays in $\mathscr{F}$, and not in clconv($\mathscr{F}$)

Then the pure-strategy $\tilde{x}^i$ is either in one of the four polyhedra.

Inner Approximate each player with a *single polyhedron*, and try to compute an *MNE*



🌈 if it terminates, it gives a *PNE*.

🙁 it might never terminate!

Algorithms

Outer Approximation

As mentioned, each player i has a feasible region $\mathscr{F}_i$ given by a union of polyhedra.

$$\mathscr{F}_i = \left\{ x^i : \begin{array}{cc} A^i x^i \leq b^i & \\ z^i = M^i x^i + q^i & \\ 0 \leq x^i_j \perp z^i_j \geq 0, & \forall j \in \mathscr{P}^i \end{array} \right\}$$

We can iteratively build up each feasible region $\mathscr{F}_i$ by adding the complementarity equations $j \in \mathscr{P}^i$

We start with just the common constraints

$$\mathscr{O}^i_0 = \{A^i x^i \leq b^i;\ x^i \geq 0;\ z^i \geq 0\}$$

Algorithms

Outer Approximation

We start with just the common constraints

$$\mathcal{O}_0^i = \{A^i x^i \leq b^i;\ x^i \geq 0;\ z^i \geq 0\}$$

VAR. SELECTION We select a complementarity id $c_j \in \mathscr{P}_i$ to be added to the approximation

$$\mathcal{O}_1^i = \text{clconv}\{\{\mathcal{O}_0 \ \cap \ x_j = 0\} \cup \{\mathcal{O}_0 \ \cap \ z_j = 0\}\}$$

NODE SELECTION We solve the node. If it is *infeasible*, then we backtrack or select a different complementarity (*MIP restarts*)

We check with a *similar rationale* of the other algorithms if an *MNE* is also an *MNE* for the original game.


Complexity
NASPs
Convexification of Games
Algorithms

Computations (PNEs)

NASPs

	EQ (s)	WINS	NO_EQ (s)	WINS	ALL (s)	TL	SOLVED
FullEnumeration	29,08	6	0,12	82	120,21	9	140
InnerApproximation	13,23	61	0,23	0	51,33	34	149
OuterApproximation	86,60	0	78,08	0	719,28	55	94

More or less the typical figure “we do better” than the others

Computations (PNEs)

NASPs

	EQ (s)	WINS	NO_EQ (s)	WINS	ALL (s)	TL	SOLVED
FullEnumeration	7,25	13	0,12	83	328,23	27	122
Combinatorial Heuristic	1,01	52	1,08	1	1,05	0	149

More or less another typical figure "we do better" than the others

The software is already available on GitHub

It consists of more than 7k lines of codes:

- Command line interface
- Standardized with C++ best practises
- Models, abstracts, and solves *LCPs, Stackelberg Games, Nash Games,* and *NASPs*
- Builds like a library that can be integrated in third-party projects
- Supports explicit modeling for energy trade markets
- To come: integration with other MIP solvers (SCIP, CPlex, ...)

An Open Source Solver

Mixed Integer Programming *(MIP)*

- Extend powerful algorithmic arsenal and developed *polyhedral tools*

Algorithmic Game Theory *(AGT)*

- *Using MIP* arsenal to model *complex interactions between agents*
- Convexification for Games

This Work

Bridging MIP and AGT

Applications

- Environmentally efficient energy markets with CO_2-caps
- An Open Source Solver