

CANADA EXCELLENCE RESEARCH



DATA SCIENCE FOR REAL-TIME DECISION-MAKING

## When Nash meets Stackelberg

Joint work with M. Carvalho, F. Feijoo, A. Lodi, S. Sankaranarayanan



![](_page_0_Picture_8.jpeg)

![](_page_0_Picture_10.jpeg)

### Wait a second!

![](_page_2_Picture_0.jpeg)

What has Game Theory to do with Integer Programming and why should we care?

A 60 seconds pitch.

### Why should we care?

 $max_x \quad c^{\mathsf{T}}x$ 

- $Ax \leq b$  $x \in \{0,1\}^n$
- E.g., a retailer building its products portfolio

### Why should we care?

 $max_{x} \quad c^{\mathsf{T}}x + x^{\mathsf{T}}Qy$  $Ax \le b$  $x \in \{0,1\}^{n}$ 

E.g., a retailer building its products portfolio

Extends typical OR problems to multi-agent settings

Fairness of algorithms and solutions

What has it to do with IP?

Broad topic for this talk, but some hints given later on...

# $max_{y} \quad d^{\mathsf{T}}x + x^{\mathsf{T}}Qy$ $Ey \leq f$ $y \in \{0,1\}^{k}$

### E.g., another retailer

![](_page_5_Picture_0.jpeg)

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![](_page_5_Picture_2.jpeg)

DATA SCIENCE FOR REAL-TIME DECISION-MAKING

## When Nash meets Stackelberg

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![](_page_5_Picture_7.jpeg)

![](_page_5_Picture_8.jpeg)

![](_page_5_Picture_10.jpeg)

![](_page_6_Picture_0.jpeg)

![](_page_6_Picture_1.jpeg)

### Consider a Bagel Shop

![](_page_7_Picture_0.jpeg)

### I usually make a case for MTL Bagels...

**Coronavirus** 

#### Coronavirus

### Macron calls for Covid vaccine exports from EU to be controlled

## a fresh crisis with exports row

### EU threatens to block Covid vaccine exports amid AstraZeneca shortfall

**Coronavirus** 

EU could block millions of Covid vaccine doses from entering UK

How EU's floundering vaccine effort hit

## theguardian

![](_page_8_Picture_9.jpeg)

![](_page_9_Picture_0.jpeg)

### Consider a Drug

![](_page_10_Picture_0.jpeg)

## *Fpizer* produces and sells its *Drug* in a market in order to profit

![](_page_11_Picture_0.jpeg)

### And competes with Giovanni & Giovanni Hence, they play a simultaneous game with the Drug

![](_page_11_Picture_2.jpeg)

![](_page_12_Picture_0.jpeg)

### Canada taxes their drugs And regulates exports/imports of the drug

![](_page_12_Picture_2.jpeg)

![](_page_12_Picture_3.jpeg)

![](_page_13_Picture_0.jpeg)

Canada regulates the market Playing a sequential game with the Drug companies

![](_page_13_Picture_2.jpeg)

Sequential Stackelberg Game

![](_page_13_Picture_4.jpeg)

### Canada

![](_page_14_Figure_1.jpeg)

### Canada competes with the UK The countries play another simultaneous game among themselves

![](_page_14_Picture_3.jpeg)

![](_page_14_Picture_4.jpeg)

### We call this Nash Game Among Stackelberg Leaders (NASP)

What if....

![](_page_15_Picture_3.jpeg)

Drug companies are instead energy producers, insurance companies, ...

![](_page_16_Picture_0.jpeg)

#### The Barolo Chapel by Sol LeWitt and David Tremlett

### Definitions and Algorithms A brief view into our work

### Background

### Stackelberg Game

(Stackelberg, 1934; Candler and Norto, 1977) In many cases, at least  $\mathcal{NP}$ -hard

Y of polyhedra

- The feasible set for the leader is often *non-convex*, and *non-connected* 
  - Basu et al. (2020) prove that under certain assumption,  $\mathcal{F}$  is a union

### Background

#### Nash Equilibrium

(Nash. 1950, 1951)

Then,  $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n)$  is a Mixed Nash Equilibrium (MNE) iff No player has an *incentive to deviate* from its equilibrium strategy  $\bar{x}^{l}$ given the other player choices  $\bar{x}^{-i}$ 

PNE: "play" multiple strategies where probabilities sum up to 1

Each player solves an optimization problem  $P^i$  depending on its decisions  $x^i$  and the one of other players  $x^{-i}$ 

When each agent plays a *mutually optimal strategy*  $x^i$  wrt the strategies  $x^{-i}$  of all the other players, then we have a so-called Nash Equilibrium

MNE: "play" a single strategy with probability 1

![](_page_19_Picture_0.jpeg)

### Some Results

![](_page_19_Picture_2.jpeg)

![](_page_19_Picture_3.jpeg)

### Complexity

![](_page_20_Picture_2.jpeg)

### Algorithms

![](_page_20_Picture_4.jpeg)

Complexity

![](_page_21_Picture_1.jpeg)

**Theorem** (Carvalho, D., Feijoo, Lodi, Sankaranarayanan, 2019)

Given a NASP with 2 leaders and 1 follower each, the followers solve a linear program and the leaders have linear objectives:

- It is  $\Sigma_2^p$ -hard to decide if the game has an *MNE* 

\_ It is  $\Sigma^p_2$ -hard to decide if the game has an PNE even if all leaders'

feasible regions are bounded

- If each player feasible region is *bounded*, then there exists an *MNE* 

The problem is not PLS-complete as many other equilibrium problems!

### There are *three* fundamental complexity theorems for *NASPs*

![](_page_21_Picture_11.jpeg)

### Complexity

![](_page_22_Picture_2.jpeg)

### Algorithms

![](_page_22_Picture_4.jpeg)

Every player *i* has a non-convex feasible region  $\mathcal{F}_i$ , made of a *union of polyhedra* 

We can use Balas's to retrieve their convex-hull cl conv( $\mathcal{F}_i$ )

We solve it and find a solution  $\tilde{x}^i$  in the convex-hull, but not within any of the original polyhedra.

![](_page_23_Picture_4.jpeg)

![](_page_23_Figure_5.jpeg)

![](_page_23_Picture_6.jpeg)

We solve it and find a solution  $\tilde{x}^i$  in the convex-hull, but not within any of the *original polyhedra*.

If it was a MIP

The solution is <u>not feasible</u>. We would search for a disjunction and *cut it off!* 

MNE interpretation

Each point in cl conv $(\mathcal{F}_i)$ \ $\mathcal{F}_i$  can be expressed as a convex combination of points *strictly laying in*  $\mathcal{F}_i$ 

If points in  $\mathscr{F}_i$  are *pure strategies*, then cl conv $(\mathscr{F}_i)$   $\mathcal{F}_i$  contains *mixed strategies*!

![](_page_24_Figure_7.jpeg)

![](_page_24_Picture_8.jpeg)

### Complexity

![](_page_25_Picture_2.jpeg)

### Algorithms

![](_page_25_Picture_4.jpeg)

Algorithms

## All the following algorithms are valid for NASPs as well as for generic Stackelberg Games!

Algorithms

#### A full enumeration

For each player *i*, go for a full enumeration of its feasible region  $\mathcal{F}_i$ - Find  $\mathcal{F}_i^* = \text{cl conv}(\mathcal{F}_i)$  using Balas'

**Theorem** (Carvalho, D., Feijoo, Lodi, Sankaranarayanan, 2019)

The full enumeration algorithm terminates finitely either with a MNE or a certificate of non-existence.

Solve the Nash Game on  $\mathscr{F}^* = \mathscr{F}_1^* \times ... \times \mathscr{F}_n^*$  for  $\tilde{x} = (\tilde{x}^1, ..., \tilde{x}^n)$ 

![](_page_27_Picture_8.jpeg)

Algorithms

each player *i*. Try to find an Equilibrium

Poly  $\frac{i}{k}$ 

There exists an MNE  $\tilde{x}$ : check if any player can deviate. This can be done with a simple *MIP* by fixing  $\tilde{x}^{-i}$  and solving for  $x^i$ :

- approximation and repeat.

## Instead of enumerating all the polyhedra, start with one polyhedron k for

![](_page_28_Picture_8.jpeg)

- If no deviation exists, then <u>TERMINATE</u>. We found an *MNE* 

![](_page_28_Picture_12.jpeg)

Algorithms

approximation and repeat

- Poly  $\frac{i}{k}$

![](_page_29_Figure_16.jpeg)

Algorithms

approximation and repeat

- Poly  $\frac{i}{k}$

![](_page_30_Picture_16.jpeg)

Algorithms

approximation and repeat

![](_page_31_Figure_3.jpeg)

Algorithms

#### Combinatorial Heuristic

polyhedra.

an MNE

*[* if it terminates, it gives a PNE. it might never terminate!

### If we are interested only in *PNEs*, then the equilibrium $\tilde{x}$ strictly lays in $\mathcal{F}$ , and not in $\operatorname{clconv}(\mathcal{F})$

Then the pure-strategy  $\tilde{x}^i$  is either in one of the four

![](_page_32_Figure_8.jpeg)

![](_page_32_Figure_9.jpeg)

![](_page_32_Figure_10.jpeg)

![](_page_32_Picture_11.jpeg)

Algorithms

#### **Outer** Approximation

polyhedra.

the complementarity equations  $j \in \mathscr{P}^i$ 

We start with just the common constraints  $\mathcal{O}_0^i = \{ A^i x^i \le b^i; \ x^i \ge 0; \ z^i \ge 0 \}$ 

As mentioned, each player i has a feasible region  $\mathcal{F}_i$  given by a union of

$$\mathcal{F}_{i} = \left\{ \begin{aligned} & A^{i}x^{i} \leq b^{i} \\ x^{i}: & z^{i} = M^{i}x^{i} + q^{i} \\ & 0 \leq x^{i}_{j} \perp z^{i}_{j} \geq 0, \end{aligned} \right. \quad \forall j \in \mathcal{P}^{i} \right\}$$

We can iteratively build up each feasible region  $\mathcal{F}_i$  by adding

Algorithms

#### Outer Approximation

$$\mathcal{O}_0^i = \{A^i x^i \le b^i;$$

**VAR. SELECTION** to be added to the approximation

 $\mathcal{O}_1^i = \operatorname{clconv}\left\{\left\{\mathcal{O}_0 \cap x_i = 0\right\} \cup \left\{\mathcal{O}_0 \cap z_i = 0\right\}\right\}$ 

NODE SELECTION We solve the node. If it is *infeasible*, then we backtrack or select a different complementarity (MIP) restarts)

We check with a *similar rationale* of the other algorithms if an MNE is also an MNE for the original game.

#### We start with just the common constraints

 $x^i \ge 0; \ z^i \ge 0$ 

We select a complementarity id  $c_i \in \mathscr{P}_i$ 

### Complexity

![](_page_35_Picture_2.jpeg)

### Algorithms

![](_page_35_Picture_4.jpeg)

### Computations (PNEs)

	EQ (s)	WINS	NO_EQ (s)	WINS	ALL (s)	TL	SOLVED
FullEnumeration	29,08	6	0,12	82	120,21	9	140
InnerApproximation	13,23	61	0,23	0	51,33	34	149
OuterApproximation	86,60	0	78,08	0	719,28	55	94

### NASPs

More or less the typical figure "we do better" than the others

![](_page_36_Picture_4.jpeg)

### Computations (PNEs)

	EQ (s)	WINS	NO_EQ (s)	WINS	ALL (S)	TL	SOLVED
FullEnumeration	7,25	13	0,12	83	328,23	27	122
Combinatorial Heuristic	1,01	52	1,08	1	1,05	0	149

### NASPs

More or less another typical figure "we do better" than the others

![](_page_37_Picture_4.jpeg)

#include <chrono</pre>

The software is already available on GitHub It consists of more than 7k lines of codes:

- Command line interface

- - Stats.AlgorithmPara

JumPlayers, value: 0);

### An Open Source Solver

olved = {false};

- Standardized with C++ best practises

- Models, abstracts, and solves LCPs, Stackelberg Games, Nash Games, and NASPs

Builds like a library that can be integrated in third-party projects

Supports explicit modeling for energy trade markets

- To come: integration with other MIP solvers (SCIP, CPlex, ...)

### Mixed Integer Programming (MIP)

- Extend powerful algorithmic arsenal and developed *polyhedral tools* 

The Horizon

Bridging MIP and AGT

### Applications

- Environmentally efficient energy markets
- Health systems
- An Open Source Solver

### Algorithmic Game Theory (AGT)

 Using MIP arsenal to model complex interactions between agents
 Convexification for Games

### Integer Programming Games

We have a *simultaneous non-cooperative* game where *n* players are solving an optimization problem with:

-linear objectives with respect to their variables, and -bi-linear objectives with respect to other players variables

We are developing:

- Exact algorithms
- Cutting planes

$$\min_{x^{i}} \left\{ (c^{i})^{\mathsf{T}} x^{i} + \sum_{j=1, j \neq i}^{n} (x^{j})^{\mathsf{T}} Q_{j}^{i} x^{i} : x^{i} \in \mathcal{X}^{i} \right\}$$

• Enhanced separation oracles

• Milder concepts of equilibria

![](_page_41_Picture_0.jpeg)

DATA SCIENCE FOR REAL-TIME DECISION-MAKING

## Thanks!

![](_page_41_Picture_3.jpeg)

![](_page_41_Picture_4.jpeg)

TECHNOLOGICAL UNIVERSITY

![](_page_41_Picture_6.jpeg)