Mathematical Programming Games

Gabriele Dragotto
Safe Robotics Lab - 09-27-2022

SAFE ROBOTICS LABORATORY

ORFE

PRINCETON UNIVERSITY

What are MPGs?

What

Why

What are MPGs?

***D.** et al (2021)*

An ***MPG*** is a (static) **game** among n players where each **rational** player $i = 1,2,\ldots,n$ solves the optimization problem

$$\max_{x^i}\{f^i(x^i, x^{-i}) : x^i \in \mathcal{X}^i\}$$

The payoff function for i

$$f^i(x^i, x^{-i}) : \prod_{j=1}^{n} \mathcal{X}^j \to \mathbb{R}$$

Is **parametrized in** x^{-i}

The set of actions for i

$$\mathcal{X}^i$$

$$\max_{x^i} \{ f^i(x^i, x^{-i}) : x^i \in \mathscr{X}^i \}$$

The payoff function for i

$$f^i(x^i, x^{-i}) : \prod_{j=1}^{n} \mathscr{X}^j \to \mathbb{R}$$

is **parametrized in** x^{-i}

The set of actions for i

$$\mathscr{X}^i$$

The choices of i's opponents affect its payoff

However, they do not affect i's actions

$$\max_{x^i} \{ f^i(x^i, x^{-i}) : x^i \in \mathscr{X}^i \}$$

Action Representation

Each player's actions are represented with **an arbitrary** set $\mathscr{X}^i$

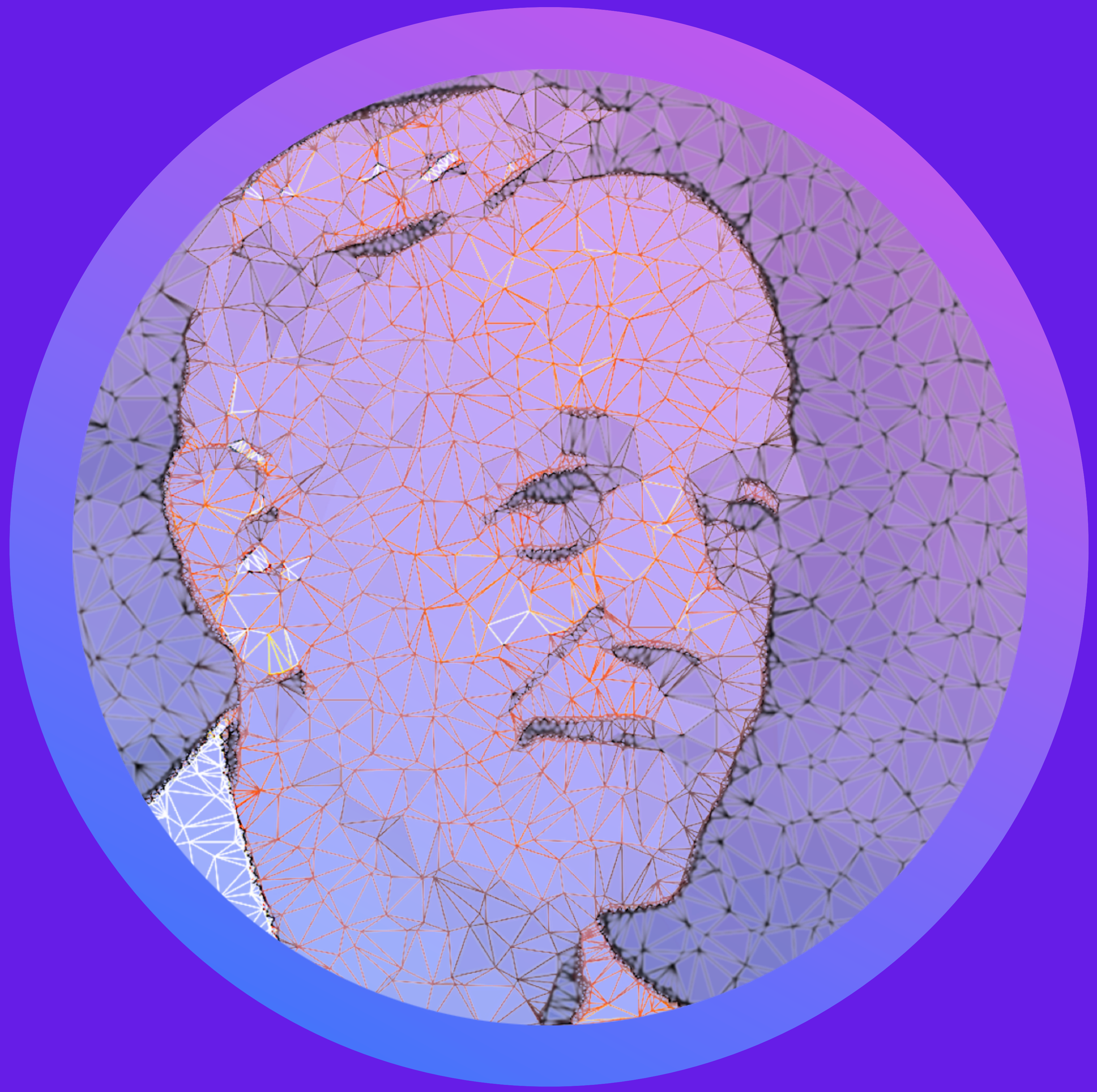
Modeling Requirements

In many applications, $\mathscr{X}^i$ may include a **complex set of operational requirements**

Language and Objectives

MPGs provide a **unified framework** to represent games from both AGT and Optimization

Equilibria as Solutions



A profile $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n)$ *— with* $\bar{x}^i \in \mathcal{X}^i$ *for any* i *— is a Pure Nash Equilibrium (***PNE***) if*

$$f^i(\bar{x}^i, \bar{x}^{-i}) \geq f^i(\hat{x}^i, \bar{x}^{-i}) \quad \forall \hat{x}^i \in \mathcal{X}^i$$

Does at least **one exist**? **How hard** is it to **compute** one?

How do we compute an NE, if any? And how do we **select one** when multiple equilibria exist?

How **efficient** is this NE?

A Few Examples

Integer Programming Games, or games among parametrized Integer Programs

Bilevel Programming and simultaneous games, specifically for energy

Network Formation Games, cost-sharing games for critical infrastructure development

A Few Examples

Integer Programming Games, or games among parametrized Integer Programs

Bilevel Programming and simultaneous games, specifically for energy

Network Formation Games, cost-sharing games for critical infrastructure development

🧙🏾‍♂️ 🧚🏼‍♂️

Open 2 Convenience Stores

$$\max_{x^1} \quad 6x_1^1 + x_2^1$$

$$\text{s.t.} \quad 3x_1^1 + 2x_2^1 \leq 4$$

$$x^1 \in \{0,1\}^2$$

Their products **interact**!

$$\max_{x^1} \quad 6x_1^1 + x_2^1 - 4x_1^1 x_1^2 + 6x_2^1 x_2^2$$

$$\text{s.t.} \quad 3x_1^1 + 2x_2^1 \leq 4$$

$$x^1 \in \{0,1\}^2$$

$$\max_{x^2} \quad 4x_1^2 + 2x_2^2 - x_1^2 x_1^1 - x_2^2 x_2^1$$

$$\text{s.t.} \quad 2x_1^2 + 3x_2^2 \leq 4$$

$$x^2 \in \{0,1\}^2$$

Knapsack Games (Carvalho et al., 2022)

☀️⚡

Energy

Carvalho, ***D.***, Lodi, Feijoo, Sankaranarayanan (2020)

Energy

Carvalho, ***D.***, Lodi, Feijoo, Sankaranarayanan (2020)

🧙 and 🧚

want to change life

Carvalho, ***D.***, Lodi, Feijoo, Sankaranarayanan (2020)

They want to sell bagels for a living

WizardMount Bagels©

Simultaneous Game

St Fairy Bagels©

Magicville taxes their bagels

Since ovens are *polluting* the city’s air

WizardMount Bagels©

Simultaneous Game

St Fairy Bagels©

Sequential
“Stackelberg” Game

Simultaneous
Game

WizardMount Bagels©

St Fairy Bagels©

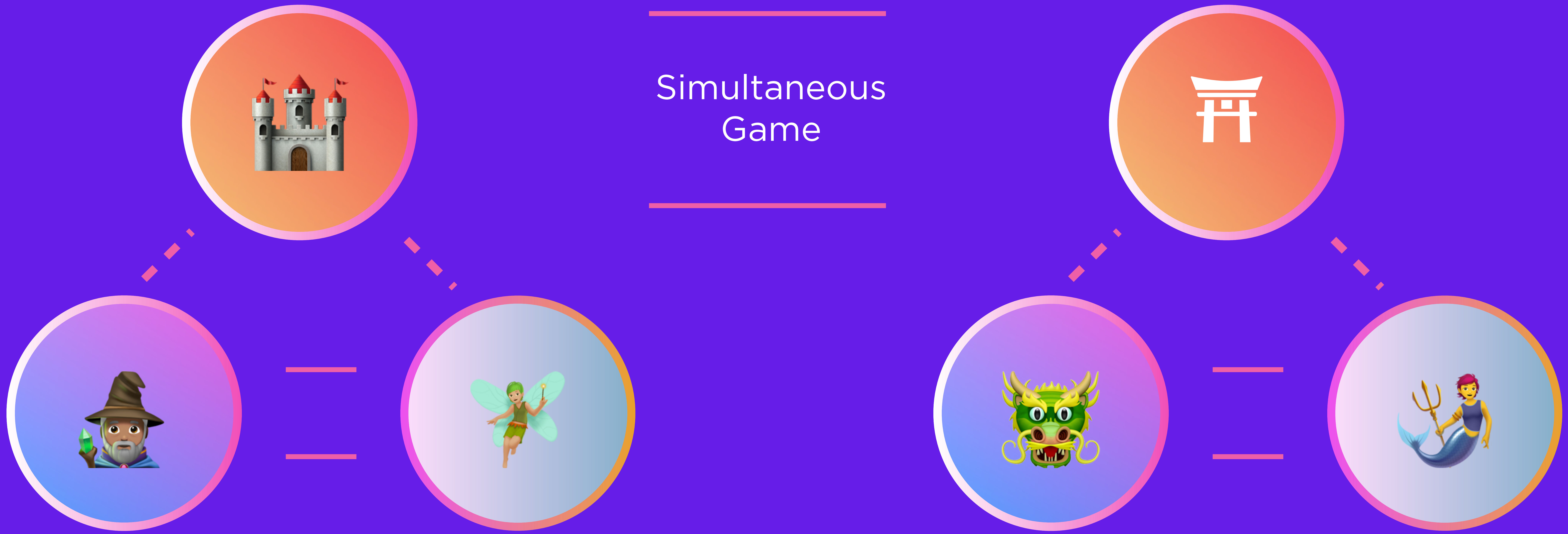
Magicville

Simultaneous Game

Witchtown

Magicville

Witchtown



Cities can import, export (or block imports and exports) of bagels
Tax their producers

We define them as Nash Games among Stackelberg Players (*NASPs*)

We define them as Nash Games among Stackelberg Players (*NASPs*)

We define them as Nash Games among Stackelberg Players (*NASPs*)

Bagels are units of energy

We define them as Nash Games among Stackelberg Players (*NASPs*)

Bagels are units of energy

Cities are regulatory agencies

Some Results

Complexity

It is Σ_2^p-hard to determine a Nash equilibrium

Algorithms

A **full enumeration scheme**, and an inner **approximation scheme**

Insights

Energy market tests, with **Chilean-Argentinean case study**

Magicville

Reformulate each Stackelberg game as a single-level Optimization problem

Magicville

$$\max_{x^i}\{(c^i)^\top x^i + (x^{-i})^\top C^i x^i : x^i \in \mathscr{F}^i\}$$

The reformulated feasible region includes the KKT for the followers' problems

$$\mathcal{F}^i = \left\{ \begin{array}{l} A^i x^i \le b^i \\ z^i = M^i x^i + q^i \\ x^i \ge 0, z^i \ge 0 \end{array} \right\} \bigcap_{j \in \mathcal{C}^i} (\{z^i_j = 0\} \cup \{x^i_j = 0\}).$$

Are leaders (countries) further reducing their emission if they optimize the income from a carbon-tax?

Does trade among countries under a carbon-tax reduce emissions?

Are leaders (countries) further reducing their emissions if they maximize an income from a carbon-tax?

It depends on what source energy producers use (i.e., coal vs solar).
In general, **no.**

Does trade among countries under a carbon-tax reduce emissions?

Are leaders (countries) further reducing their emission if they optimize the income from a carbon-tax?

It depends on what source energy producers use (i.e., coal vs solar).
In general, **no.**

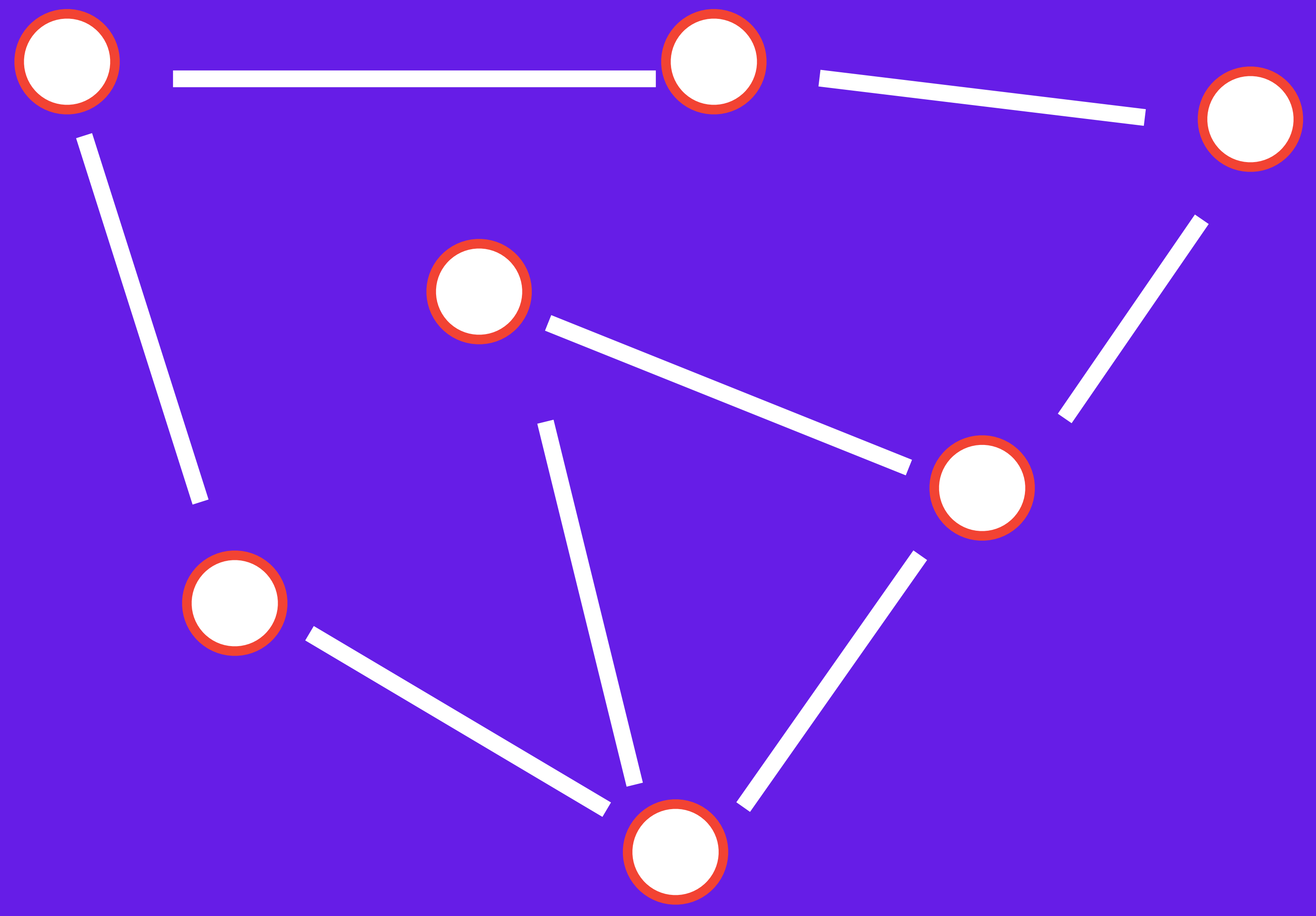
Does trade among countries under a carbon-tax reduce emissions?

Since trade is about money, the **intuitive answer is no.**
However, we found that countries with large quantities of clean energy can fulfil the need of countries with fossil fuel, **thus reducing the overall emissions.**

💻🌎

Network Formation

Network Formation Game



(Chen and Roughgarden, 2006;
Anshelevich, et al., 2008;
Nisan et al., 2008)

Given a graph $G = (V, E)$:

- Any $(h, l) \in E : h, l \in V$ has a cost $c_{hl} \in \mathbb{Z}^+$
- Player i needs to go **from** s^i **to** t^i

  Player i **has a weight** w^i

The cost of each edge is **split proportionally to each player's weight**

(Past) Research Core Questions

Modeling	How flexible are MPGs as a modelling tool?
Existence	When does at least an equilibrium exist?
Efficiency	How do different equilibria (solutions) in MPGs differ?
Algorithms	How do we compute and select equilibria?
Insights	Do equilibria promote socially-beneficial outcomes and provide insights?

Speculative Research Questions

Dynamics	How does the interaction happen over time? Can it lead to equilibria?
Uncertainty	What if some parameters of the game are unknown?
Learning	Could we learn something about the agents' (parametric) strategic behavior?
Applications	Can IPGs be applied to Safety and Robotics?

How?

~~What~~ ~~are **Mathematical Programming Games**~~

~~Why~~ ~~do we need them, some **applications**, and **core research questions**~~

How do we use and ***solve*** them in practice

How?

How do we use and solve them in practice

ZERO Regrets

Optimizing over equilibria in **Integer Programming Games**

(Dragotto and Scatamacchia, 2021)

Cut-And-Play

Computing Nash equilibria via **Convex Outer Approximations**

(Carvalho et al., 2021)

The ZERO Regrets Algorithm

Joint work with **Rosario Scatamacchia** (Politecnico di Torino, Italy)

How

Integer Programming Games

*Integer Programming Games (**IPGs**) are* MPGs where each player $i = 1,2,\ldots,n$ solves

(Köppe et al., 2011)

$$\max_{x^i}\{u^i(x^i, x^{-i}) : x^i \in \mathcal{X}^i\},\ \mathcal{X}^i := \{A^i x^i \leq b^i, x^i \in \mathbb{Z}^m\}$$

There is **common knowledge of rationality**, thus each player is **rational** and there is **complete information**,

However, there are a few issues:

Selection

Not all Nash equilibria were created equal
i.e., **Price of Stability (PoS) and Anarchy (PoA)**

Tractability

Existence

Restrictive assumptions on the game's structure to guarantee the existence/tractability

Methodology

Lack of a general-purpose methodology to compute and mostly **select** equilibria

⇒ **No general methodology, no broad use of IPGs.**

⇛ **No general methodology, no broad use of IPGs.**

The core motivation behind ZERO Regrets:

Provide a **general-purpose and efficient** *algorithmic and theoretical* framework to **compute, select and enumerate** Nash equilibria in IPGs.

				Type of NE			
	General	Enumer.	Select	PNE	NE	Approx	Limitations
ZERO Regrets	✅	✅	✅	✅	✅	✅	**Most efficient, selection, existence, enumeration**
Koeppe et al. (2011)	✅	✅	❌	✅	❌	❌	No (practical) algorithm
Sagratella (2016)	✅	✅	❌	✅	❌	❌	Convex payoffs
Del Pia et al. (2017)	❌	❌	❌	✅	❌	❌	Problem-specific (unimodular)
Carvalho, D., Lodi, Sankaranarayanan (2020)	✅	❌	❌	❌	✅	❌	Bilinear payoffs
Cronert and Minner (2021)	✅	✅	❌	❌	✅	❌	No selection, expensive, existence?
Carvalho et al. (2022)	✅	❌	❌	❌	✅	✅	No selection/enumeration, existence?
Schwarze and Stein (2022)	✅	✅	❌	✅	❌	❌	Expensive Branch-and-Prune

Contributions

Theoretical

Polyhedral characterization: strategic interaction in terms of inequalities, polyhedral closures

Algorithms

Cutting plane algorithm: computes, ***selects***, enumerates **Nash equilibria.**

Practical

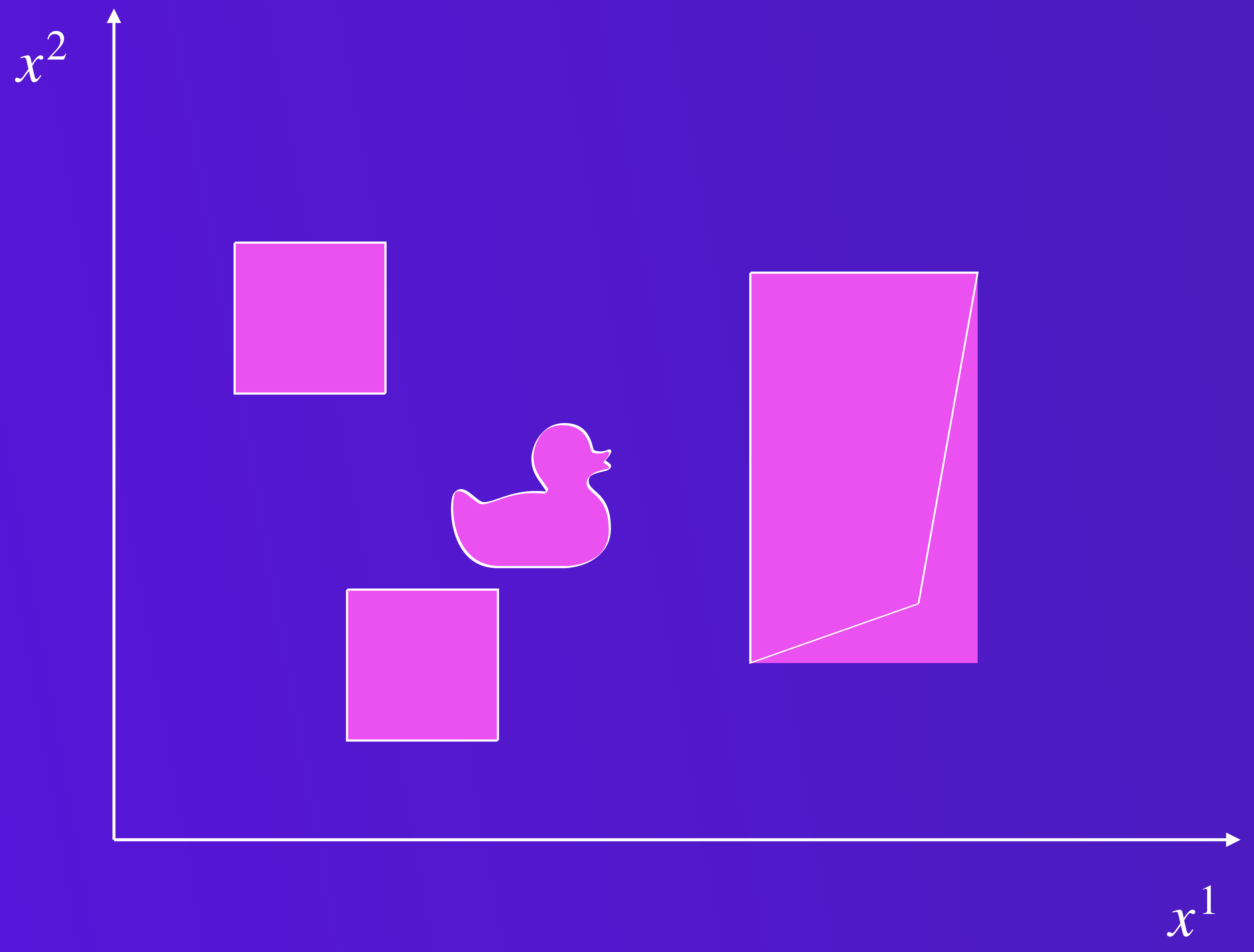
Several **applications** and methodological problems

Algorithmic Idea

A Lifted Space for Equilibria

A Lifted Space for Equilibria

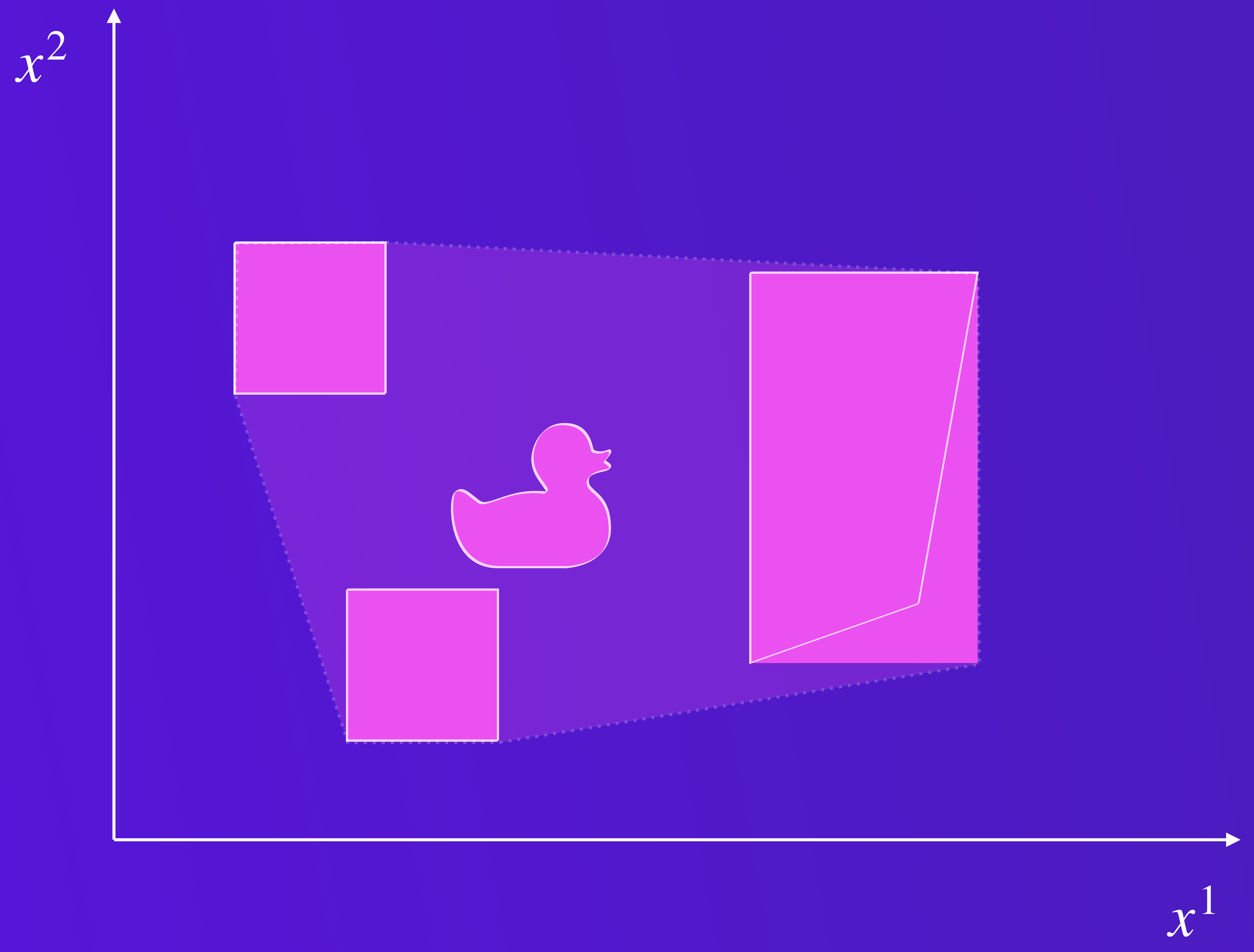
$$\max_{x^i}\{u^i(x^i, x^{-i}) : x^i \in \mathcal{X}^i\},\ \mathcal{X}^i := \{A^i x^i \le b^i, x^i \in \mathbb{Z}^m\}$$



$$\prod_{i=1}^{n} \mathcal{X}^i$$

A Lifted Space for Equilibria

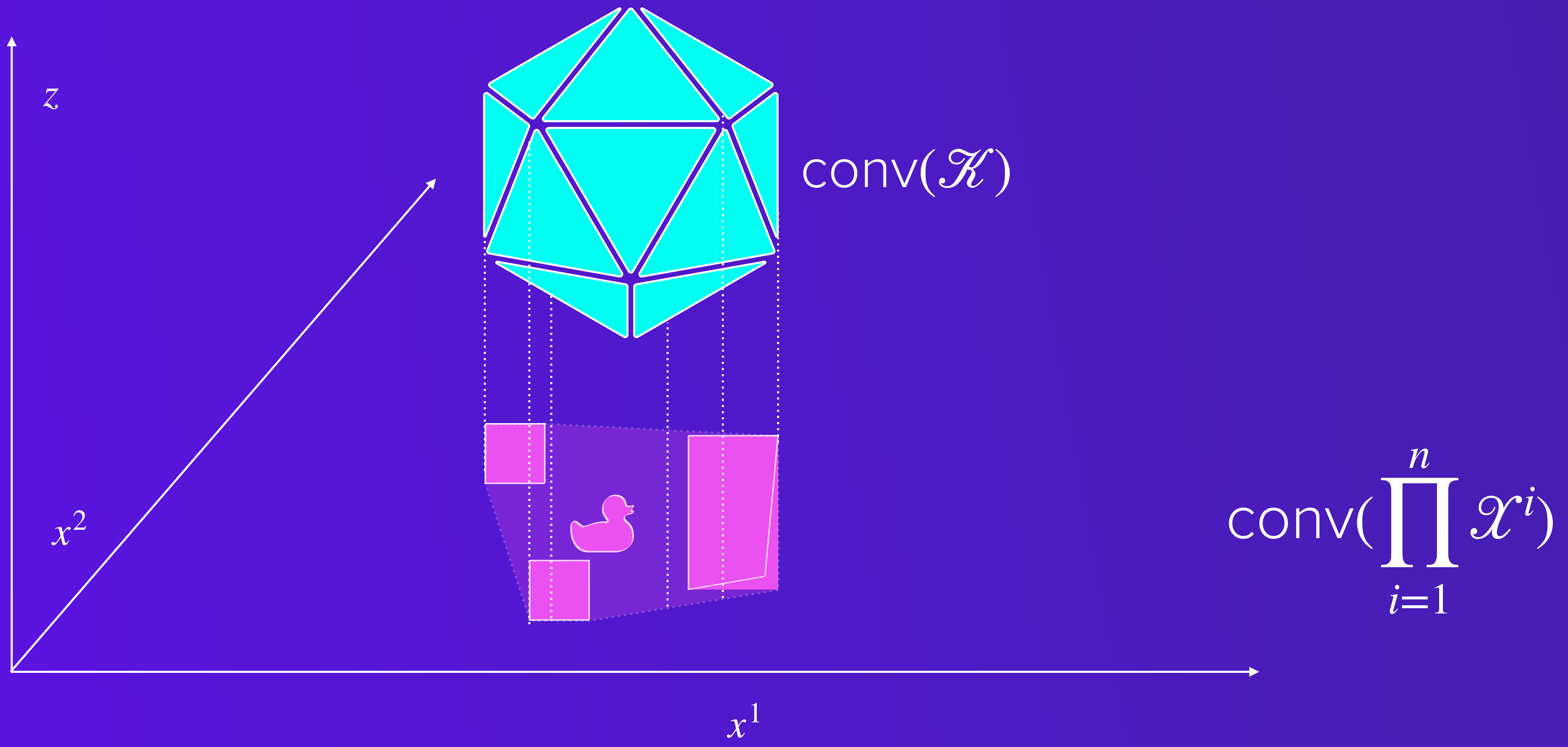
$$\max_{x^i} \{u^i(x^i, x^{-i}) : x^i \in \mathcal{X}^i\}, \; \mathcal{X}^i := \{A^i x^i \leq b^i, x^i \in \mathbb{Z}^m\}$$



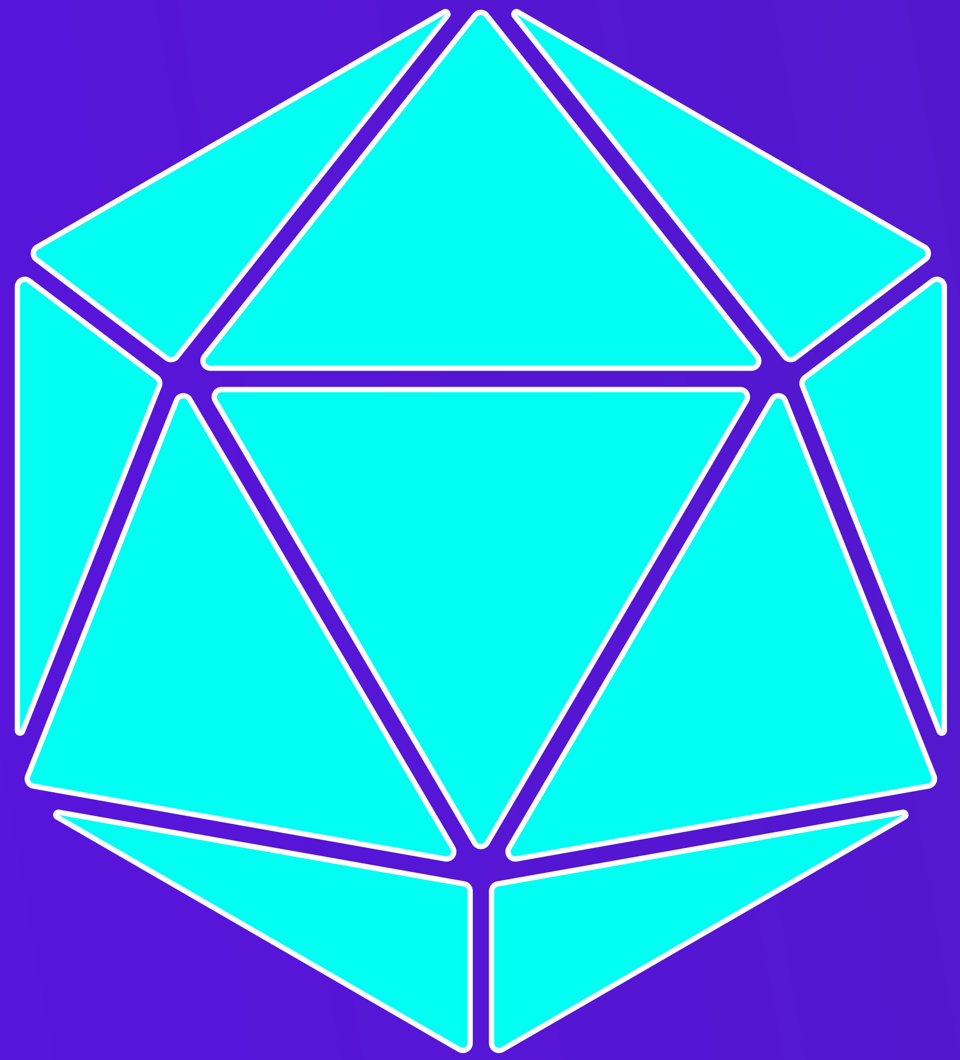
$$\text{conv}\left(\prod_{i=1}^{n} \mathcal{X}^i \right)$$

A Lifted Space for Equilibria

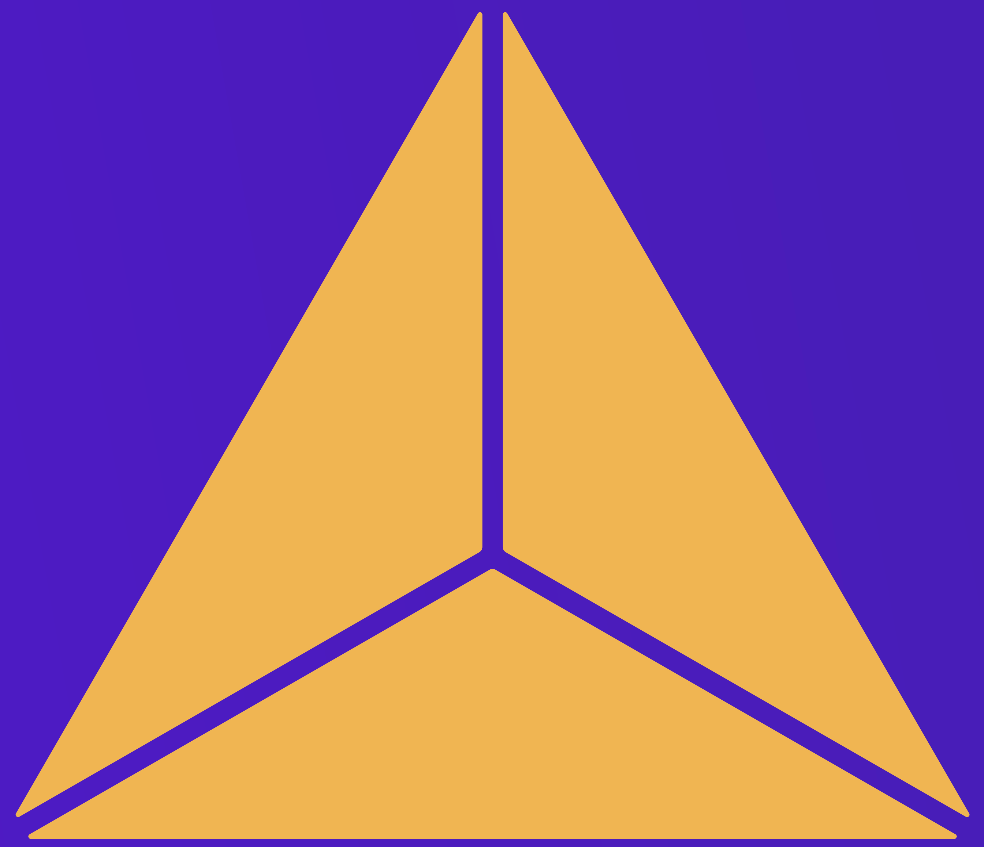
$$\mathscr{K} = \{(x^1, \ldots, x^n, z) : x^i \in \mathscr{X}^i, (x, z) \in \mathscr{L}\}$$



A Lifted Space for Equilibria



$\supseteq$



conv($\mathscr{K}$)

conv("Nash Equilibria")

The Goal

The Goal

Given an IPG $f : \prod_i \mathcal{X}^i \to \mathbb{R}$, compute the Nash equilibrium maximizing f

The Goal

Given an IPG $f : \prod_i \mathscr{X}^i \to \mathbb{R}$, compute the Nash equilibrium maximizing f

The Idea

The Goal

Given an IPG $f : \prod_i \mathcal{X}^i \to \mathbb{R}$, compute the Nash equilibrium maximizing f

The Idea

Start from an approximation of conv$(\mathcal{K})$ refine it until optimizing f over it yields a point $(\bar{x}, \bar{z})$ so that with $x \in$ conv("Nash Equilibria")

Inequalities

Equilibrium Inequality

An inequality is an **equilibrium inequality** *if it is valid for*
$x \in$ *conv("Nash Equilibria")*

Namely, equilibrium inequalities *cut off feasible strategies for some players* but **never equilibrium profiles!**

$$u^i(\tilde{x}^i, x^{-i}) \leq u^i(x^i, x^{-i}) \quad \forall \tilde{x}^i \in \mathcal{X}^i$$

Applications

Applications

	Applications	Baselines	Select	Enumer.	Improvement
Knapsack Game	Packing, Assortment Optimization	Carvalho et al. (2021, 2022)	❌	❌	N.A.
Network Formation Games	Network design, the Internet, cloud infrastructure	Chen and Roughgarden (2006), Anshelevich, et al. (2008), Nisan et al. (2008)	✅	❌	N.A.
Facility Location Games	Retail, cloud service provisioning	Cronert and Minner (2021)	✅	❌	>50x
Quadratic Integer Games	Mostly methodological	Sagratella (2016), Schwarze and Stein (2022)	✅	✅	10x to 600x

Thanks!