Mathematical Programming Games

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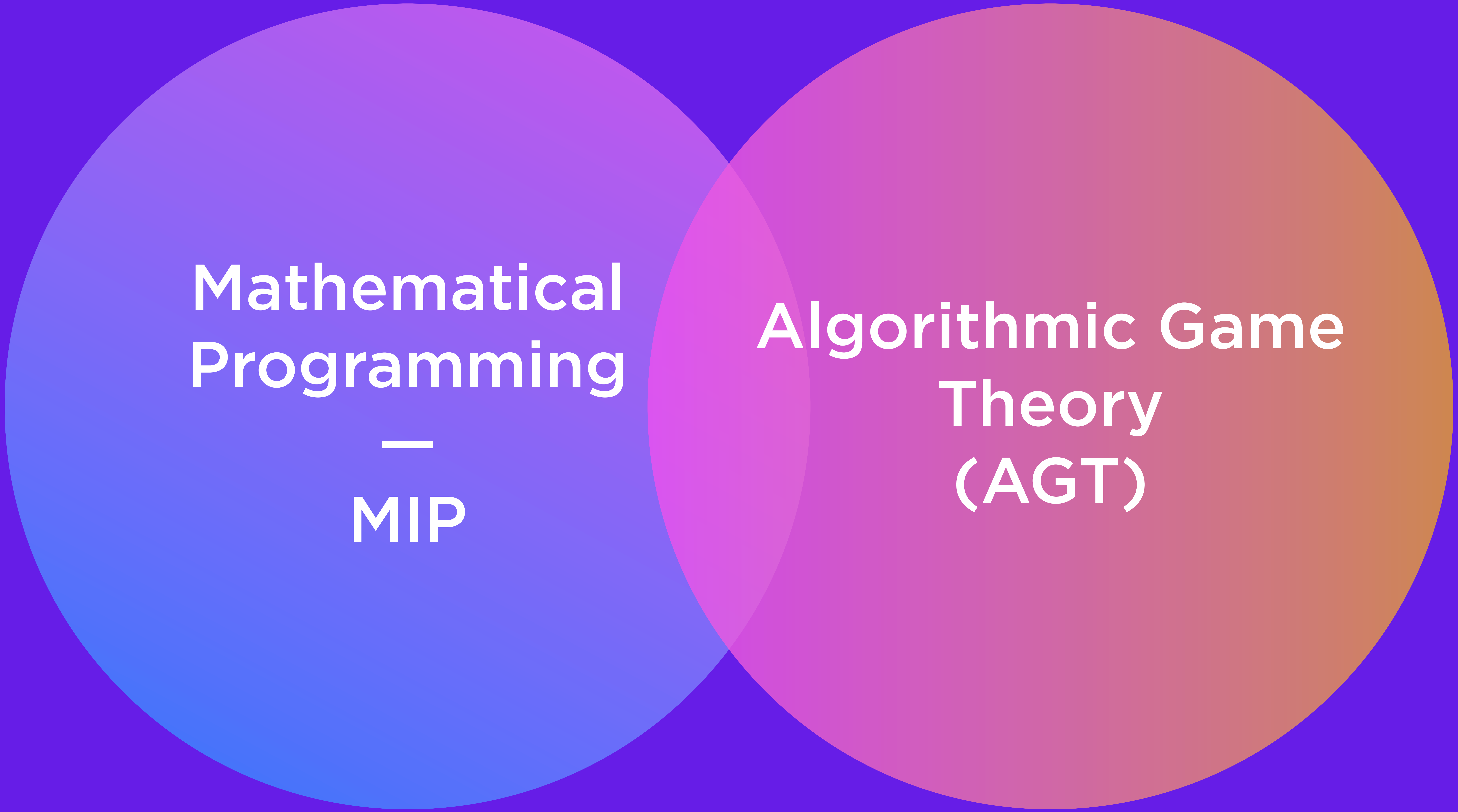
Optimization/ECE Seminar - May 17th, 2022

CANADA EXCELLENCE RESEARCH CHAIR

DATA SCIENCE FOR REAL-TIME DECISION-MAKING

PRINCETON UNIVERSITY



Mathematical Programming

—

MIP

Algorithmic Game Theory (AGT)

Mathematical Programming Games (MPGs)

A Brief Overview of This Talk

What are **Mathematical Programming Games**

Why do we need them, some **applications**, and **core research questions**

How do we ***solve*** them in practice

What are MPGs?

What

Why

What are MPGs?

***D.** et al (2021)*

An ***MPG*** is a (static) **game** among n players where each **rational** player $i = 1,2,\ldots,n$ solves the optimization problem

$$\max_{x^i} \{ \underline{f^i(x^i, x^{-i})} : x^i \in \underline{\mathcal{X}^i} \}$$

The payoff function for i

$$f^i(x^i, x^{-i}) : \prod_{j=1}^{n} \mathcal{X}^j \to \mathbb{R}$$

Is **parametrized in** x^{-i}

The set of actions for i

$$\mathcal{X}^i$$

$$\max_{x^i}\{\underline{f^i(x^i, x^{-i})} : x^i \in \underline{\mathscr{X}^i}\}$$

The payoff function for i

$$f^i(x^i, x^{-i}) : \prod_{j=1}^{n} \mathscr{X}^j \to \mathbb{R}$$

is **parametrized in x^{-i}**

The set of actions for i

$$\mathscr{X}^i$$

The choices of i's opponents affect its payoff

However, they do not affect i's actions

$$\max_{x^i} \{ f^i(x^i, x^{-i}) : x^i \in \mathscr{X}^i \}$$

Action Representation

Each player's actions are represented with **an arbitrary** set $\mathscr{X}^i$

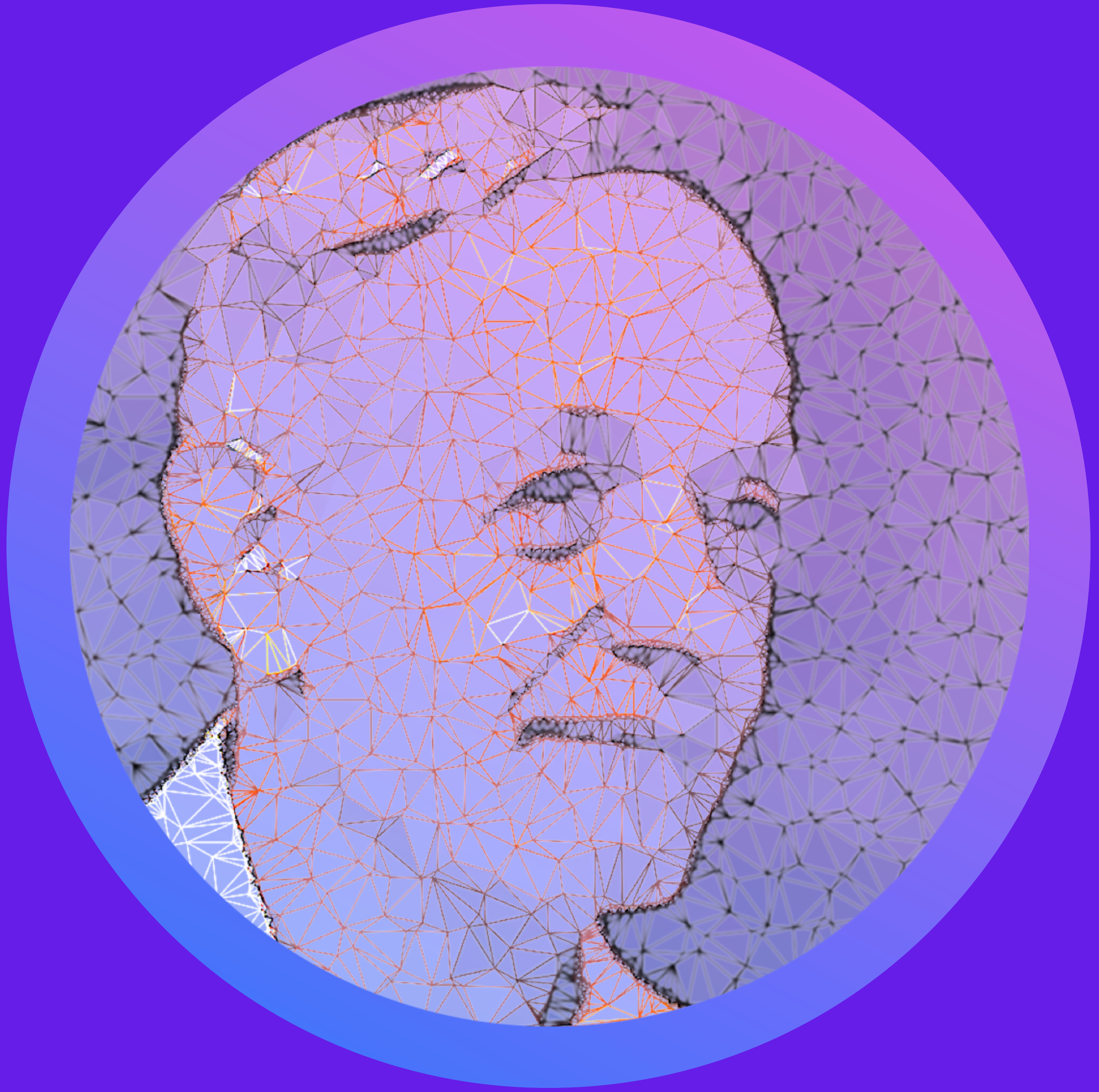
Modeling Requirements

In many applications, $\mathscr{X}^i$ may include a **complex set of operational requirements**

Language and Objectives

MPGs provide a **unified framework** to represent games from both AGT and Optimization

Equilibria as Solutions



A profile $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^n)$ *— with* $\bar{x}^i \in \mathcal{X}^i$ *for any* i *— is a Pure Nash Equilibrium (***PNE***) if*

$$f^i(\bar{x}^i, \bar{x}^{-i}) \geq f^i(\hat{x}^i, \bar{x}^{-i}) \quad \forall \hat{x}^i \in \mathcal{X}^i$$

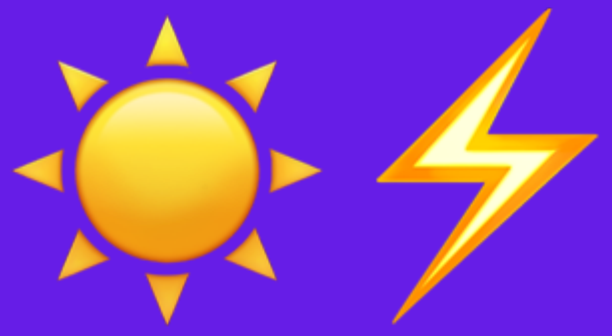
Does at least **one exist**? **How hard** is it to **compute** one?

How do we compute an NE, if any? And how do we **select one** when multiple equilibria exist?

How **efficient** is this NE?

A Few Examples

Integer Programming Games, or games among parametrized Integer Programs



Bilevel Programming and simultaneous games, specifically for energy

Network Formation Games, cost-sharing games for critical infrastructure development

A Few Examples

🧙 🧚 **Integer Programming Games**, or games among parametrized Integer Programs

☀️ ⚡ **Bilevel Programming and simultaneous games,** specifically for energy

💻 🌎 **Network Formation Games,** cost-sharing games for critical infrastructure development

🧙🏾‍♂️ 🧚🏼‍♂️

Open 2 Convenience Stores

$$\begin{aligned} \max_{x^1} \quad & 6x_1^1 + x_2^1 \\ \text{s.t.} \quad & 3x_1^1 + 2x_2^1 \leq 4 \\ & x^1 \in \{0,1\}^2 \end{aligned}$$

Their products **interact**!

$$\max_{x^1} \quad 6x_1^1 + x_2^1 - 4x_1^1x_1^2 + 6x_2^1x_2^2$$
$$\text{s.t.} \quad 3x_1^1 + 2x_2^1 \leq 4$$
$$x^1 \in \{0,1\}^2$$

$$\max_{x^2} \quad 4x_1^2 + 2x_2^2 - x_1^2x_1^1 - x_2^2x_2^1$$
$$\text{s.t.} \quad 2x_1^2 + 3x_2^2 \leq 4$$
$$x^2 \in \{0,1\}^2$$

Knapsack Games (Carvalho et al., 2022)

☀️⚡

Energy

Carvalho, ***D.***, Lodi, Feijoo, Sankaranarayanan (2020)

SolarCorp Inc.

Simultaneous Game

Hydro Inc.

🇨🇦

Canada taxes and regulates the production

☀️

SolarCorp Inc.

Simultaneous Game

🌊

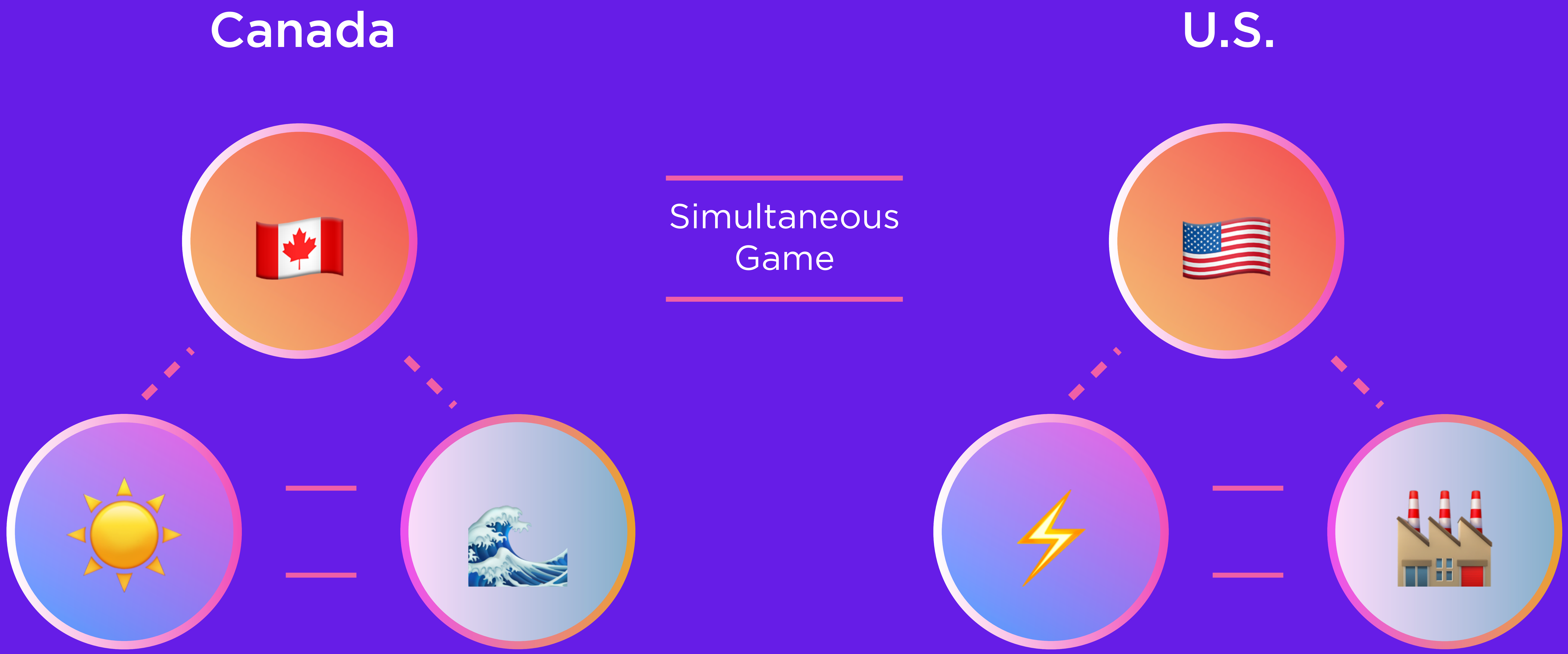
Hydro Inc.

Sequential
“Stackelberg” Game

Simultaneous
Game

SolarCorp Inc.

Hydro Inc.

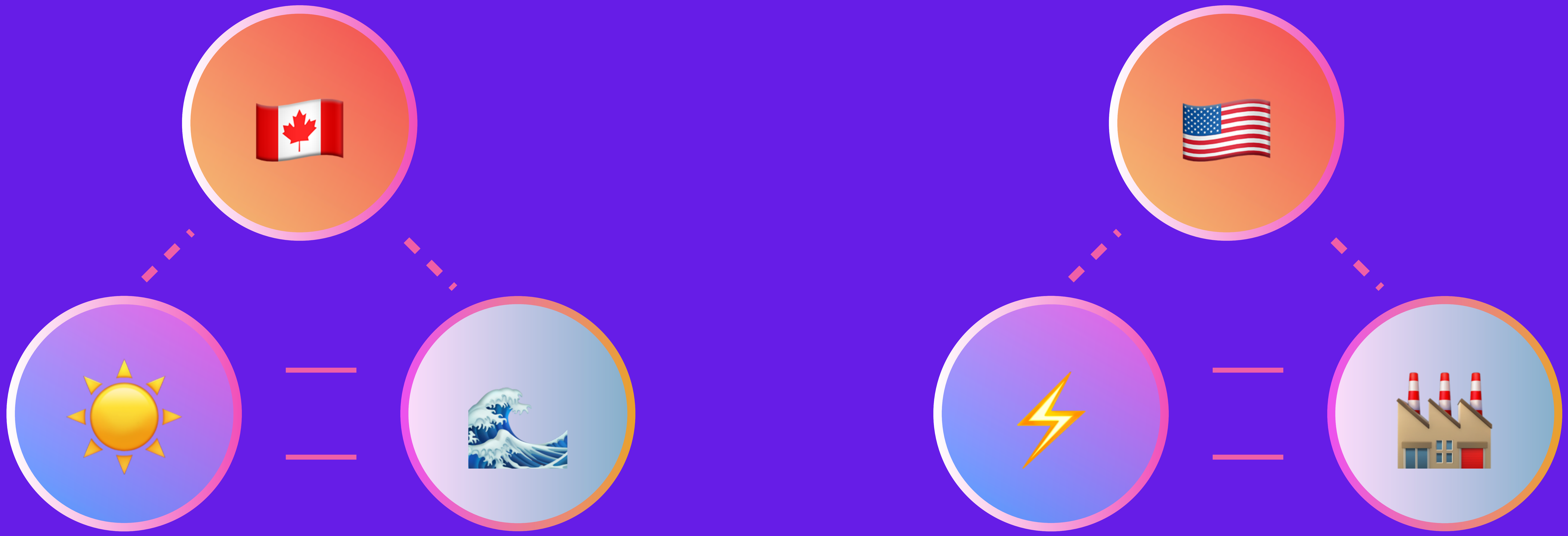


Canada

U.S.

Simultaneous Game

This is a simultaneous game among bilevel (i.e., sequential) programs



Canada

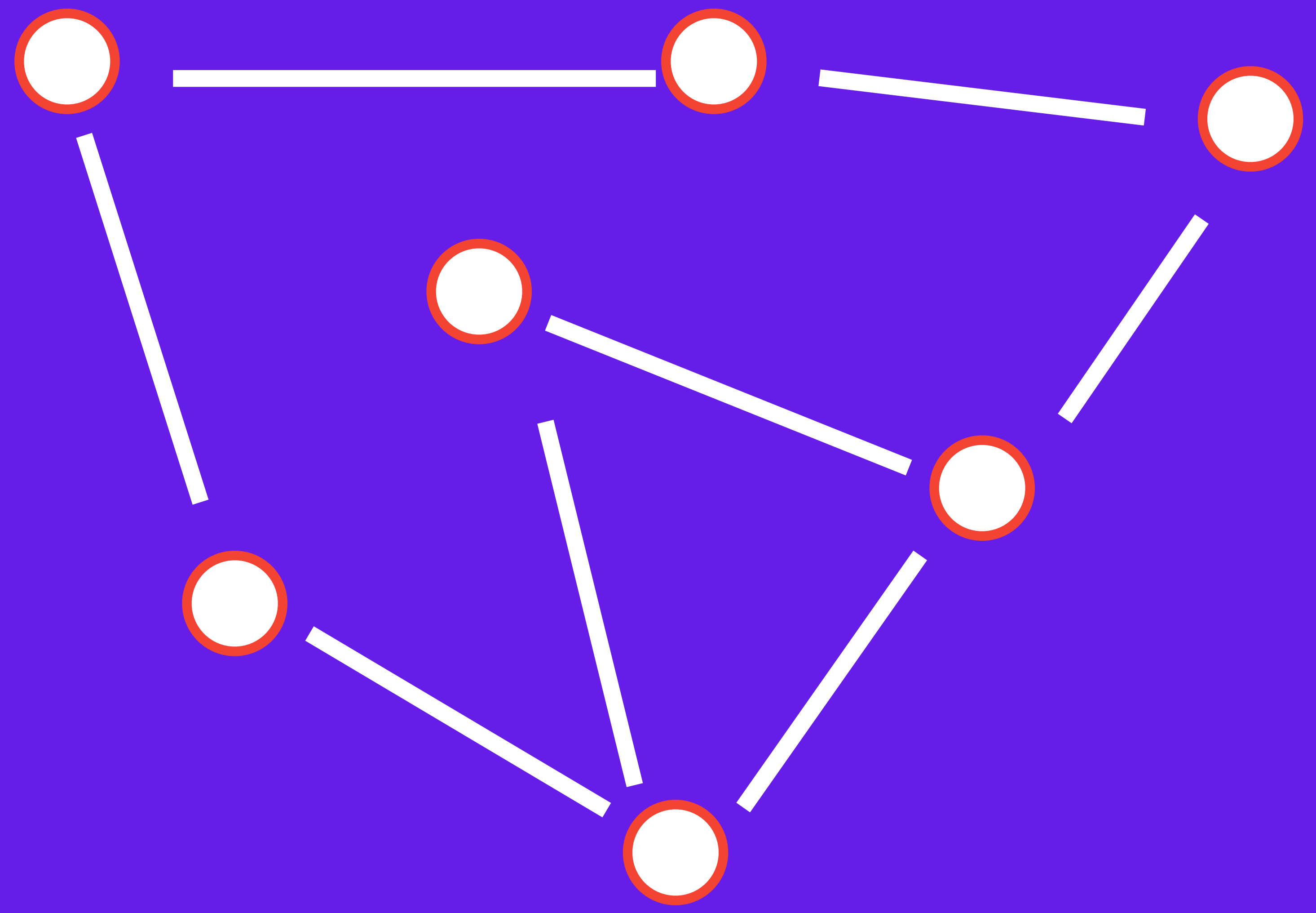
U.S.

Each $\mathscr{X}^i$ includes the optimality conditions of each “follower” (i.e., producer)

💻🌎

Network Formation

Network Formation Game



(Chen and Roughgarden, 2006;
Anshelevich, et al., 2008;
Nisan et al., 2008)

Given a graph $G = (V, E)$:

- Any $(h, l) \in E : h, l \in V$ has a cost $c_{hl} \in \mathbb{Z}^+$
- Player i needs to go **from** s^i **to** t^i

  Player i **has a weight** w^i

The cost of each edge is **split proportionally to each player's weight**

Core Research Questions

Modeling	Can MPGs model real-world problems?
Existence	When does at least an equilibrium exist?
Efficiency	How do different equilibria (solutions) in MPGs differ?
Algorithms	How do we compute and select equilibria?
Insights	Do equilibria promote socially-beneficial outcomes and provide insights?

How?

~~What~~ ~~are **Mathematical Programming Games**~~

~~Why~~ ~~do we need them, some **applications**, and **core research questions**~~

How do we use and ***solve*** them in practice

How?

How do we use and solve them in practice

ZERO Regrets

Optimizing over equilibria in **Integer Programming Games**

(Dragotto and Scatamacchia, 2021)

The ZERO Regrets Algorithm

Joint work with **Rosario Scatamacchia** (Politecnico di Torino, Italy)

How

Integer Programming Games

Integer Programming Games (***IPGs***) *are* MPGs where each player $i = 1,2,\ldots,n$ solves

(Köppe et al., 2011)

$$\max_{x^i}\{u^i(x^i, x^{-i}) : x^i \in \mathcal{X}^i\},\ \mathcal{X}^i := \{A^i x^i \leq b^i, x^i \in \mathbb{Z}^m\}$$

There is **common knowledge of rationality**, thus each player is **rational** and there is **complete information**,

Why IPGs?

They extend traditional **resource-allocation tasks and combinatorial optimization** problems to a multi-agent setting

Indivisible quantities, **fixed production costs** and **logical disjunctions** often require discrete variables
(i.e., *Bikhchandani and Mamer (1997)*)

Energy — Gabriel et al. (2013), David Fuller and Çelebi (2017)
Supply Chain — Anderson et al. (2017)
Assortment-Price competitions — Federgruen and Hu (2015)
Kidney Exchange Problems — Carvalho et al. (2017)
Cybersecurity

However, there are a few issues:

Selection

Not all Nash equilibria were created equal
i.e., **Price of Stability (PoS) and Anarchy (PoA)**

Tractability

Existence

Restrictive assumptions on the game's structure to guarantee the existence/tractability

Methodology

Lack of a general-purpose methodology to compute and mostly **select** equilibria

⇒ **No general methodology, no broad use of IPGs.**

⇒ **No general methodology, no broad use of IPGs.**

The core motivation behind ZERO Regrets:

Provide a **general-purpose and efficient** *algorithmic and theoretical* framework to **compute, select and enumerate** Nash equilibria in IPGs.

				Type of NE			
	General	Enumer.	Select	PNE	NE	Approx	Limitations
ZERO Regrets	✅	✅	✅	✅	✅	✅	**Most efficient, selection, existence, enumeration**
Koeppe et al. (2011)	✅	✅	❌	✅	❌	❌	No (practical) algorithm
Sagratella (2016)	✅	✅	❌	✅	❌	❌	Convex payoffs
Del Pia et al. (2017)	❌	❌	❌	✅	❌	❌	Problem-specific (unimodular)
Carvalho, D., Lodi, Sankaranarayanan (2020)	✅	❌	❌	❌	✅	❌	Bilinear payoffs
Cronert and Minner (2021)	✅	✅	❌	❌	✅	❌	No selection, expensive, existence?
Carvalho et al. (2022)	✅	❌	❌	❌	✅	✅	No selection/enumeration, existence?
Schwarze and Stein (2022)	✅	✅	❌	✅	❌	❌	Expensive Branch-and-Prune

Contributions

Theoretical

Polyhedral characterization: strategic interaction in terms of inequalities, polyhedral closures

Algorithms

Cutting plane algorithm: computes, ***selects***, enumerates **Nash equilibria.**

Practical

Several **applications** and methodological problems

A Lifted Space for Equilibria

Lifted Space

The sets $\mathcal{K}$ and $\mathcal{E}$

Linearize u^i with some variables z and linear constraints $\mathcal{L}$.

$$\mathcal{K} = \{(x^1, \dots, x^n, z) \in \mathcal{L}, x^i \in \mathcal{X}^i \text{ for any } i = 1, \dots, n\}$$

$\text{proj}_x(\text{conv}(\mathcal{K}))$ = **all the strategy profiles**

Let $\mathcal{N} := \{x = (x^1, \dots, x^n) : x \text{ is a NE}\}$. Consider the set

$$\mathcal{E} = \{(x^1, \dots, x^n, z) \in \text{conv}(\mathcal{K}) : (x^1, \dots, x^n) \in \text{conv}(\mathcal{N})\}$$

Lifted Space

The sets $\mathcal{K}$ and $\mathcal{E}$

$$\mathcal{E} = \{(x^1, \ldots, x^n, z) \in \mathrm{conv}(\mathcal{K}) : (x^1, \ldots, x^n) \in \mathrm{conv}(\mathcal{N})\}$$

Is the so-called **Perfect Equilibrium Formulation**

Namely, optimizing a function $f : \mathcal{K} \to \mathbb{R}$ over $\mathcal{E}$ gives the **Nash equilibrium maximizing** f (for any vertex of $\mathcal{E}$)

The Goal

The Goal

Given an IPG f, compute the Nash equilibrium maximizing f

The Goal

Given an IPG and f, compute the Nash equilibrium maximizing f

The Idea

The Goal

Given an IPG and f, compute the Nash equilibrium maximizing f

The Idea

Start from $\operatorname{conv}(\mathcal{K})$ and get to some *intermediate polyhedron* over which optimizing f yields a point $(\bar{x}, \bar{z}) \in \mathcal{E}$ with $\bar{x} \in \mathcal{N}$

Inequalities

Equilibrium Inequality

An inequality is an **equilibrium inequality** *if it is valid for* $\mathcal{E}$

Namely, equilibrium inequalities *cut off feasible strategies for some players* but **never equilibrium profiles!**

$$u^i(\tilde{x}^i, x^{-i}) \leq u^i(x^i, x^{-i}) \quad \forall \tilde{x}^i \in \mathcal{X}^i$$

Are these inequalities enough for $\mathcal{E}$?

Are these inequalities enough for $\mathcal{E}$?

Yes: all the inequalities together describe precisely $\mathcal{E}$

Separating Equilibrium Inequalities

Oracle

Equilibrium Oracle

Given a point $(\bar{x}, \bar{z})$ *and* $\mathcal{E}$, *the* **equilibrium separation problem** is the task of determining that either:

$(\bar{x}, \bar{z}) \in \mathcal{E}$ **and** ✅

$(\bar{x}, \bar{z}) \notin \mathcal{E}$ **+ an equilibrium inequality**

🔮

The Equilibrium Separation Oracle

ZERO Regrets

ZERO Regrets

INPUT: An IPG Instance and a function f

OUTPUT: A PNE $\bar{x}$

```
A set of inequalities $\Phi = \{0 \leq 1\}$
While (STOP)
  $(\bar{x},\bar{z}) = \arg\max_{x^1,\dots,x^n,z} \{f(x,z) : (x,z) \in \mathrm{conv}(\mathcal{K}), \quad \Phi\}$

  If 🔮$(\bar{x},\bar{z})$ says yes: $\bar{x}$ is the PNE maximizing $f$
  Else 🔮$(\bar{x},\bar{z})$ says no:
    add at least a violated equilibrium
    inequality to $\Phi$
```

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Applications

Applications

	Applications	Baselines	Select	Enumer.	Improvement
Knapsack Game	Packing, Assortment Optimization	Carvalho et al. (2021, 2022)	❌	❌	N.A.
Network Formation Games	Network design, the Internet, cloud infrastructure	Chen and Roughgarden (2006), Anshelevich, et al. (2008), Nisan et al. (2008)	✅	❌	N.A.
Facility Location Games	Retail, cloud service provisioning	Cronert and Minner (2021)	✅	❌	>50x
Quadratic Integer Games	Mostly methodological	Sagratella (2016), Schwarze and Stein (2022)	✅	✅	10x to 600x

Knapsack Game (*KPG*)

As for Wizard and Fairy, each player solves a binary Knapsack problem with some **interaction terms** in the objective

$$\max_{x^i} \left\{ \sum_{j=1}^{m} p_j^i x_j^i + \sum_{k=1,k\neq i}^{n} \sum_{j=1}^{m} C_{k,j}^i x_j^i x_j^k : \sum_{j=1}^{m} w_j^i x_j^i \leq b^i, \mathbf{x}^i \in \{0,1\}^m \right\}$$

Knapsack Game (*KPG*)

A few facts:

- No successful attempts to **enumerate or select** equilibria in KPGs with $n > 2$ and $m > 4$ (*Cronert and Minner (2021)*)
- Carvalho et al. (2021, 2022) only compute **an MNE** with at most $n = 3, m \leq 40$
- No results on the complexity of the KPG, nor its PoS/PoA

We select PNEs with $n > 2, m > 50$
We provide "packing" equilibrium inequalities

We prove it is Σ_2^p-complete to determine if a PNE exists + the PoS/PoA are arbitrarily bad

Knapsack Game (*KPG*)

Equilibrium inequalities may also **capture specific structures** or constraint types.

Strategic Payoff Inequalities

A fact

In a packing problem, often the all-zeros strategy is feasible with objective 0

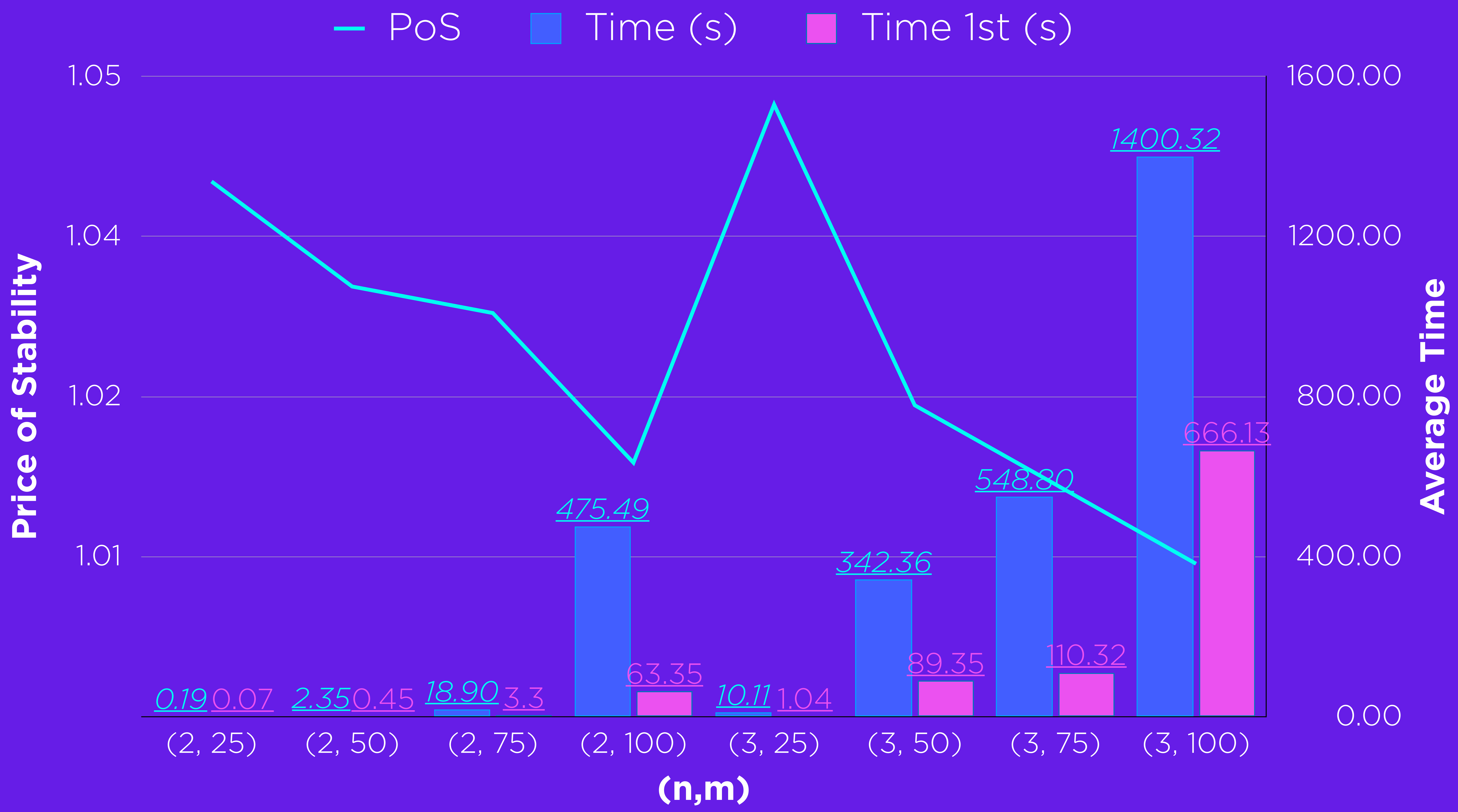
A consequence

Let $\mathcal{S}_i$ be a subset of i's opponents. If $\exists \mathcal{S}_i$ so that

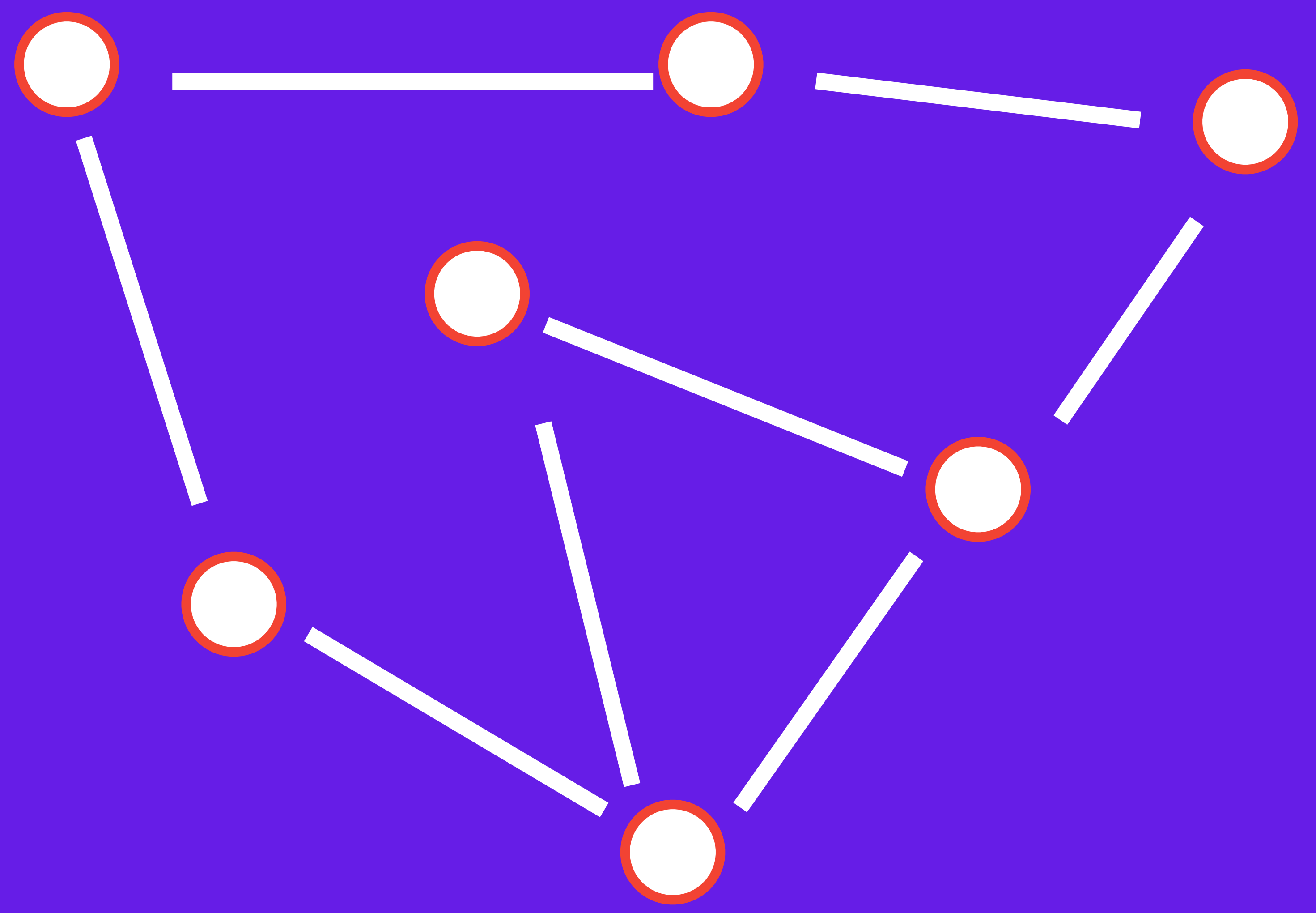
$$p_j^i + \sum_{k \in \mathcal{S}_j^i} C_{k,j}^i < 0,$$

then, $x_j^i + \sum_{k \in \mathcal{S}_j^i} x_j^k \leq |\mathcal{S}_j^i|$ is an **equilibrium inequality**.

Knapsack Game



Network Formation Game



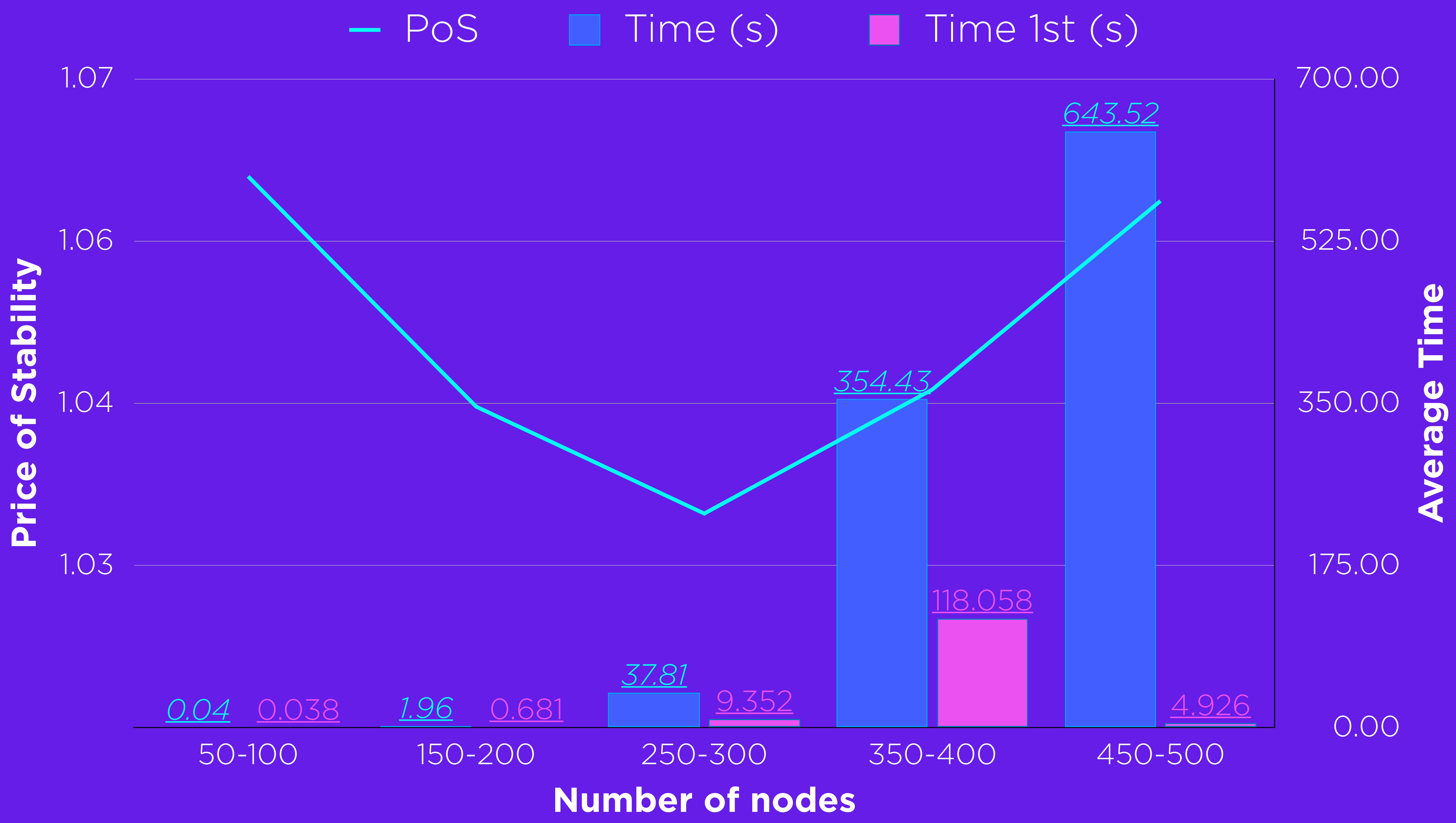
$G = (V, E)$

$$\min_{x^i}\{\sum_{(h,l)\in E}\frac{c_{hl}x^i_{hl}}{\sum_{k=1}^{n}x^k_{hl}} : x^i \in \mathcal{F}^i\}.$$

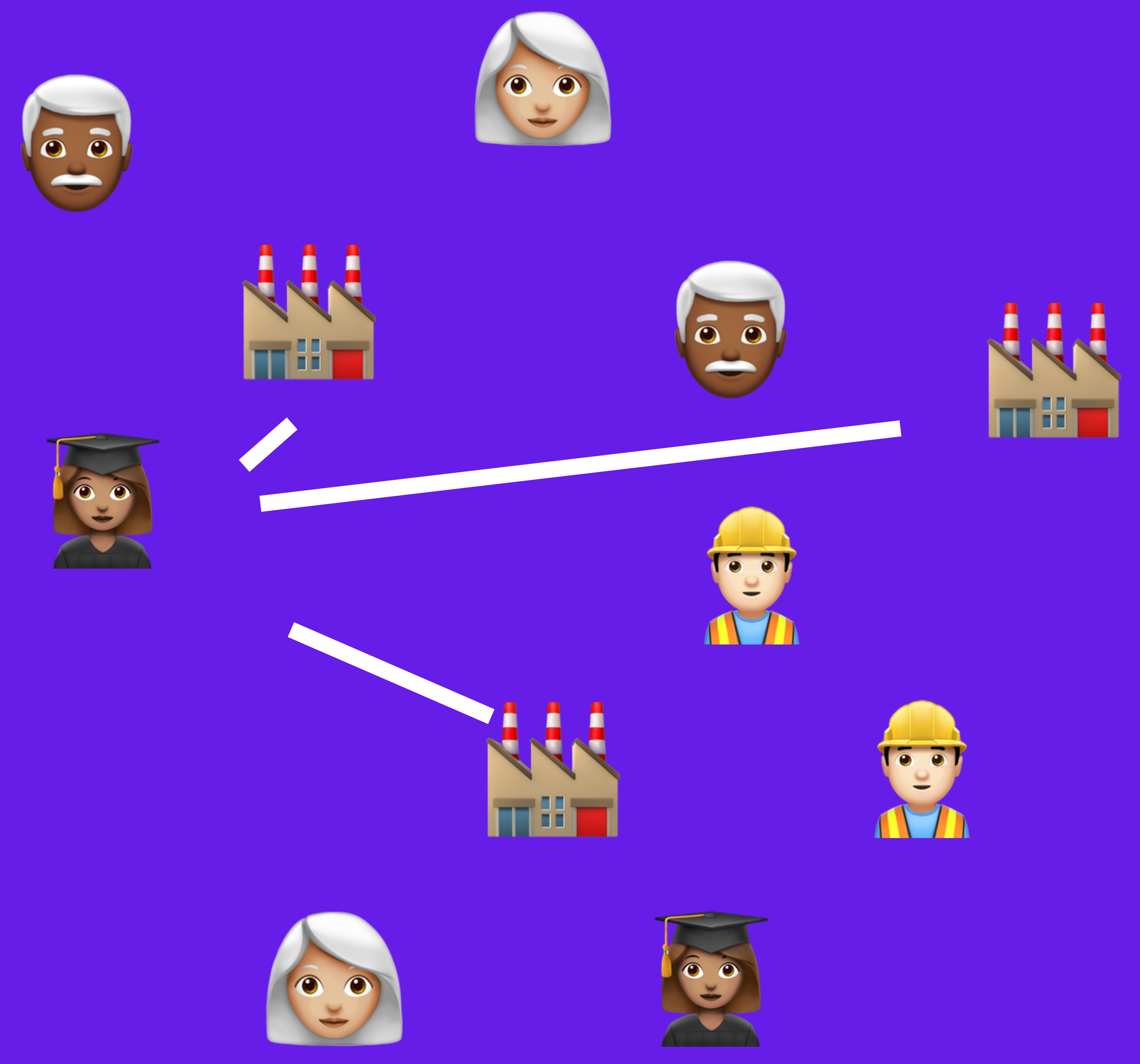
A few facts:

- No algorithms to **select** equilibria in arbitrary NFGs
- Several bounds on PoS/PoA in some specific instances
- We consider the **weighted version** with $n = 3$

Network Formation Game



Facility Location and Design Game



Aboolian et al. (2007),
Cronert and Minner (2020),

Sellers (players) compete for the demand of customers located in a given geographical area. Each player decides:

- **Where** to open its selling facilities
- **What** assortment to sell (i.e., what design)

$$\max_{x^i} \quad \sum_{j\in J} w_j \frac{\sum_{l\in L}\sum_{r\in R_l} u^i_{ljr}x^i_{lr}}{\sum_{k=1}^n \sum_{l\in L}\sum_{r\in R_l} u^k_{ljr}x^k_{lr}}$$

Share of customers' demand

$$\text{s.t.} \sum_{l\in L}\sum_{r\in R_l} f^i_{lr}x^i_{lr} \le B^i,$$

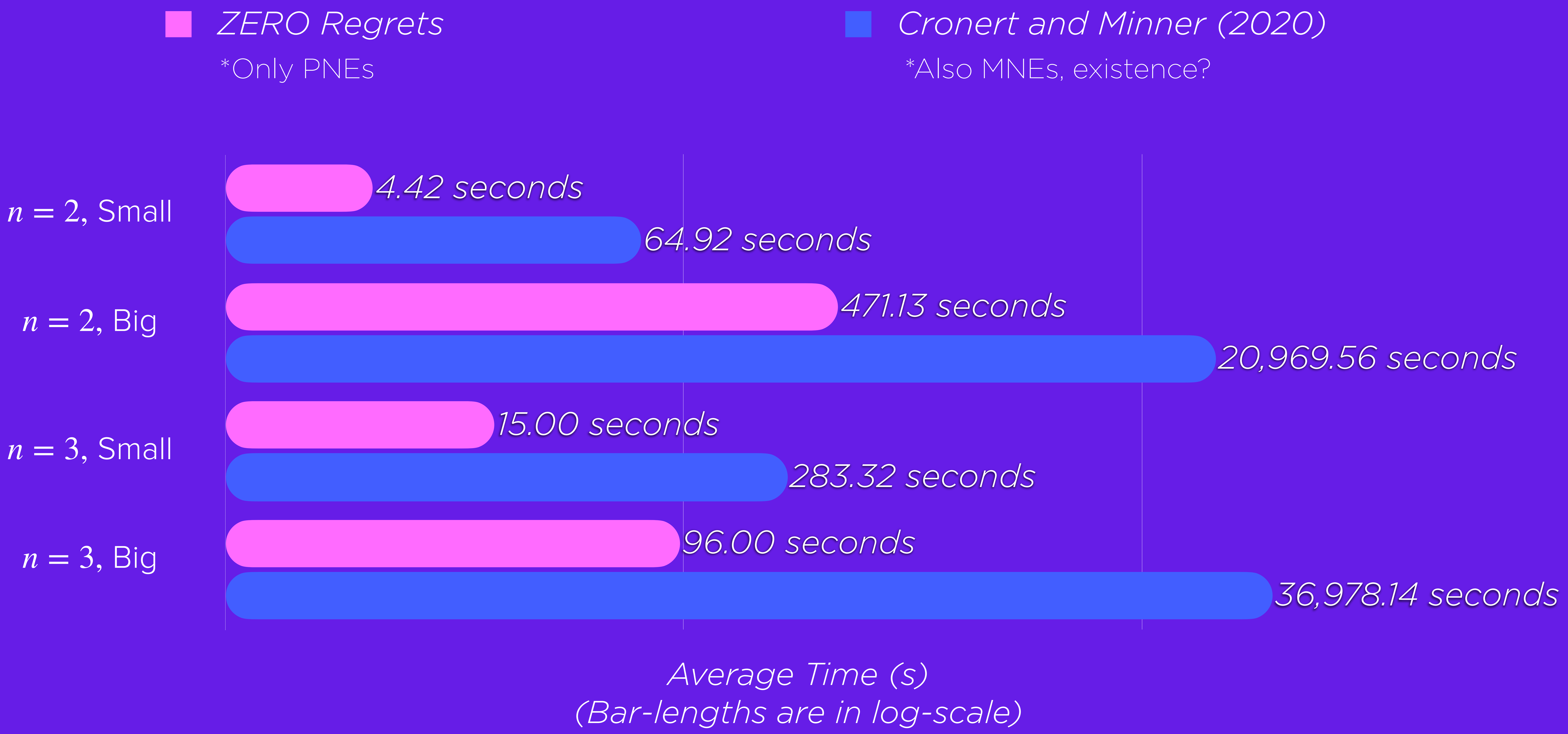
Budget

$$\sum_{r\in R_l} x^i_{lr} \le 1 \quad \forall l \in L,$$

One facility per location

$$x^i_{lr} \in \{0,1\} \quad \forall l \in L, \forall r \in R_l.$$

Facility Location and Design Game



Quadratic Integer Games

Each player i solves:

$$\min_{x^i}\{\frac{1}{2}(x^i)^\top Q^i x^i + (C^i x^{-i})^\top x^i + (c^i)^\top x^i : LB \leq x^i \leq UB,\ x^i \in \mathbb{Z}^m\}.$$

Schwarze and Stein (2022), Sagratella (2016)

	Convex Objectives	**Non-Convex Objectives**
ZERO Regrets	*413 seconds* no time-limits (1h)	*101 seconds* no time-limits (1h)
Schwarze and Stein (2022)	*64553 seconds* 13 time-limits (1h)	*65807 seconds* 13 time-limits (1h)

Remarks, Ideas, Directions

Some Remarks

In MPGs, the plausibility of the Nash equilibrium can only stem from the availability of ***efficient tools*** *to compute it.*

Optimization Framework

Scalable and flexible

Optimize over Equilibria

Beyond ZERO Regrets

MPGs

IPGs

RBGs

Finite Games

MPGs

If non-convexities are not necessarily integer:

$$\max_{x^i}\{f^i(x^i,x^{-i}) = (c^i)^\top x^i + (x^{-i})^\top C^i x^i : x^i \in \mathscr{X}^i\}$$

So-called **Reciprocally-Bilinear Games**

Margarida Carvalho, Gabriele Dragotto, Andrea Lodi, Sriram Sankaranarayanan, *The Cut and Play Algorithm: Computing Nash Equilibria via Outer Approximations*, **arXiv:2111.05726**

An MPG library

ZERO

https://github.com/ds4dm/ZERO

Gabriele Dragotto, Sriram Sankaranarayanan, Margarida Carvalho, Andrea Lodi, *ZERO: Playing Mathematical Programming Games*, **arXiv:2111.07932**

Directions

Methodology

Developments of efficient algorithms and theoretical frameworks to handle **general non-convexities**

Rational behavior through inequalities and Optimization**,** new solutions concepts

Learning the parametrized problems of each player

Practice

MPGs and applications

Fairness

Companies, governments, and in general, organizations are likely to solve optimization problems. Trade-off ***selfishness and social good***

Methodology

Practice

Fairness

Gabriele Dragotto and Rosario Scatamacchia, *The ZERO Regrets Algorithm: Optimizing over Pure Nash Equilibria via Integer Programming*, **arXiv:2111.06382**

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Roughgarden T, Tardos E (2004) Bounding the inefficiency of equilibria in nonatomic congestion games. **Games and Economic Behavior** *47(2):389–403*

Schwarze S, Stein O (2022) A branch-and-prune algorithm for discrete Nash equilibrium problems. **Optimization Online Preprint** *ID 2022-03-8836:27,*

Extra

“Princeton faculty played a major role in the development of the theory of convex optimization (both as it relates to individuals and groups). Indeed, H. Kuhn and A.W. Tucker (in the 1950s) and R.T. Rockafellar (in the 1960s) played an important role in the development of much of convex analysis. In addition, R. Gomorry laid the foundations of integer programming in the 1950s, and J. Nash, J. von Neumann, and O. Morgenstern (in the 1930s, 40s and 50s) did seminal work in the area of multi-agent, **non-cooperative optimization (i.e., game theory).”**

How do we compute an NE, if any? And how do we **select one** when multiple equilibria exist?

Lemke-Howson Generalizations

Lemke and Howson, 1964;
Rosenmüller, 1971;
Wilson, 1971;
Avis et al., 2010;
Audet et al., 2006.

Homotopy-based

Scarf, 1967.

Equilibrium Programming

Facchinei and Pang, 2003;
Sagratella, 2016;
Pang and Scutari, 2011.

Support Enumeration

Sandholm et al., 2005;
Porter et al., 2008.

MIP

Sandholm et al., 2005;
Cronert and Minner, 2021;
Carvalho et al., 2022.

Computing one is often difficult, **selecting** one is even more challenging.

A Quick Comparison

Equilibrium Programming

❌ Often $\mathscr{X}^i$ is continuous

❌ Algos: Complementarity or V.I.

❌ Global convergence?

❌ Non-convexities?

✅ Efficient in well-behaved cases

Normal/Extensive-form games

❌ No complex operational constraints

❌ Explicit (and *burdensome*) representation of action sets

✅ Popular in Game Theory literature

Their items **interact**!

$$\max_{x^1} \quad 6x_1^1 + x_2^1 - 4x_1^1x_1^2 + 3x_2^1x_2^2$$

$$\text{s.t.} \quad 3x_1^1 + 2x_2^1 \leq 4$$

$$x^1 \in \{0,1\}^2$$

$$\max_{x^2} \quad 4x_1^2 + 2x_2^2 - x_1^2x_1^1 - x_2^2x_2^1$$

$$\text{s.t.} \quad 3x_1^2 + 2x_2^2 \leq 4$$

$$x^2 \in \{0,1\}^2$$

How good is a NE? Can we select one?

The “central” authority

$$\begin{aligned} \max_{x^1} \quad & 6x_1^1 + x_2^1 - 4x_1^1x_1^2 + 6x_2^1x_2^2 \\ \text{s.t.} \quad & 3x_1^1 + 2x_2^1 \leq 4 \\ & x^1 \in \{0,1\}^2 \end{aligned}$$

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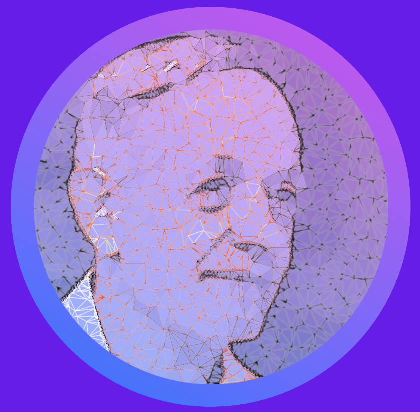
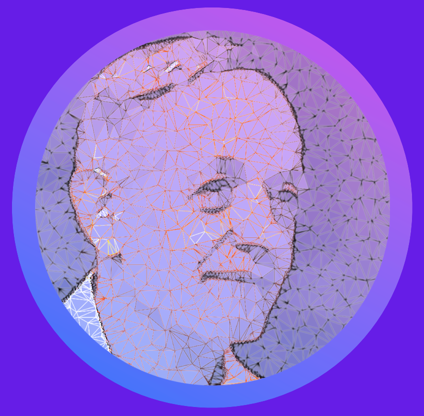
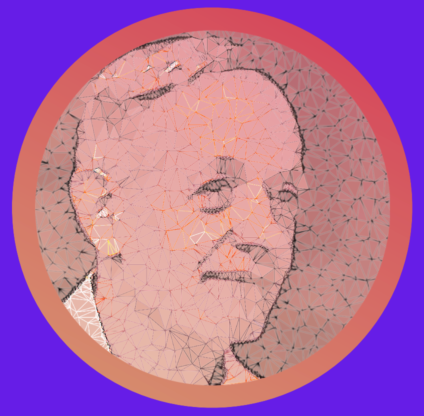


$(\bar{x}_1^1, \bar{x}_2^1) = (1,0)$ and $(\bar{x}_1^2, \bar{x}_2^2) = (1,0)$ with $W = 2 + 3 = 5$

$(\bar{x}_1^1, \bar{x}_2^1) = (1,0)$ and $(\bar{x}_1^2, \bar{x}_2^2) = (0,1)$ $W = 6 + 2 = 8$



$$= \frac{\text{Optimal Social Welfare}}{\text{"Best" NE}} = PoS$$



$$= \frac{\text{Optimal Social Welfare}}{\text{"Worst" NE}} = PoA$$

The Closure

Equilibrium Closure

THEOREM

The equilibrium closure of $\mathrm{conv}(\mathcal{K})$ *given by the set of equilibrium inequalities from before is given by:*

$$P^e := \left\{ (x,z) \in \mathrm{conv}(\mathcal{K}) \;\middle|\; \begin{array}{l} u^i(\tilde{x}^i, x^{-i}) \leq u^i(x^i, x^{-i}) \\ \forall \tilde{x} : \tilde{x}^i \in \mathcal{BR}(i, \tilde{x}^{-i}),\ i = 1, \ldots, n \end{array} \right\}$$

where $\mathcal{BR}(i, \tilde{x}^{-i})$ *are the best-responses of* $\mathbf{i}$ *given* $\tilde{x}^{-i}$. ***Then:***

- P^e is a rational polyhedron, and
- int(P^e) contains no points $(\bar{x}, \bar{z}) : \bar{x} \in \mathbb{Z}^{mn}$, and
- $P^e = \mathcal{E}$.

Equilibrium Separation Oracle

INPUT: A profile $(\bar{x}, \bar{z})$ and an IPG Instance

OUTPUT: yes or no and Φ

```
For every player $i = 1,2,\ldots,n$
    $\hat{x}^i \leftarrow \max_{x^i}\{u^i(x^i, \bar{x}^{-i}) : A^i x^i \leq b^i, x^i \in \mathbb{Z}^m\}$
    If $u^i(\hat{x}^i, \bar{x}^{-i}) > u^i(\bar{x}^i, \bar{x}^{-i})$:
        Add $u^i(\hat{x}^i, x^{-i}) \leq u^i(x^i, x^{-i})$ to $\Phi$

If $\Phi$ is empty: return yes
Else: return no and $\Phi$
```

MNEs and Approximate Equilibria

Other Equilibria

Equilibrium Inequality

An inequality is an **equilibrium inequality** *if it is valid for* $\mathcal{E}$

This includes, by definition, **any pure strategy appearing in at least an MNE**

Adding a **small quantity** γ to any equilibrium inequality ⇒ γ**-Pure Nash equilibrium**

$$u^i(\tilde{x}^i, x^{-i}) - \gamma \leq u^i(x^i, x^{-i})$$

Knapsack Game

(n, m, t)	EQIne	Time (s)	PoS	#TL
(2, 25, A)	14.67	0.06	1.04	0/3
(2, 25, B)	17.33	0.12	1.02	0/3
(2, 25, C)	29.33	0.39	1.06	0/3
(2, 50, A)	20.00	0.21	1.02	0/3
(2, 50, B)	26.67	0.51	1.01	0/3
(2, 50, C)	72.67	6.34	1.08	0/3
(2, 75, A)	38.00	0.60	1.00	0/3
(2, 75, B)	100.67	8.35	1.02	0/3
(2, 75, C)	112.67	47.75	1.08	0/3
(2, 100, A)	25.33	0.76	1.01	0/3
(2, 100, B)	205.33	220.42	1.01	0/3
(2, 100, C)	697.33	1205.29	1.05	2/3

(n, m, t)	#EQIne	Time (s)	PoS	#TL
(3, 25, A)	31.00	0.21	1.01	0/3
(3, 25, B)	44.00	0.33	1.02	0/3
(3, 25, C)	91.00	29.78	1.26	0/3
(3, 50, A)	95.00	18.39	1.03	0/3
(3, 50, B)	206.00	626.45	1.01	1/3
(3, 50, C)	148.00	382.24	-	0/3
(3, 75, A)	64.00	4.65	1.02	0/3
(3, 75, B)	278.00	982.97	1.01	1/3
(3, 75, C)	173.00	658.77	-	1/3
(3, 100, A)	261.00	1200.65	1.00	2/3
(3, 100, B)	479.00	1800.00	-	3/3
(3, 100, C)	184.00	1200.31	-	2/3

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Network Formation Game

(|V|, |E|)	EQIne	T (s)	T-1st	PoS	TL
(50, 99)	6.00	0.04	0.04	1.12	0/3
(100, 206)	2.33	0.05	0.04	1.00	0/3
(150, 308)	6.00	0.64	0.25	1.01	0/3
(200, 416)	11.67	3.28	1.11	1.06	0/3
(250, 517)	64.67	63.50	16.07	1.02	0/3
(300, 626)	21.00	12.11	2.64	1.04	0/3
(350, 730)	19.00	13.92	7.42	1.01	0/3
(400, 822)	248.67	694.95	228.69	1.08	1/3
(450, 934)	394.67	1199.98	2.61	1.11	2/3
(500, 1060)	35.67	87.07	7.25	1.00	0/3

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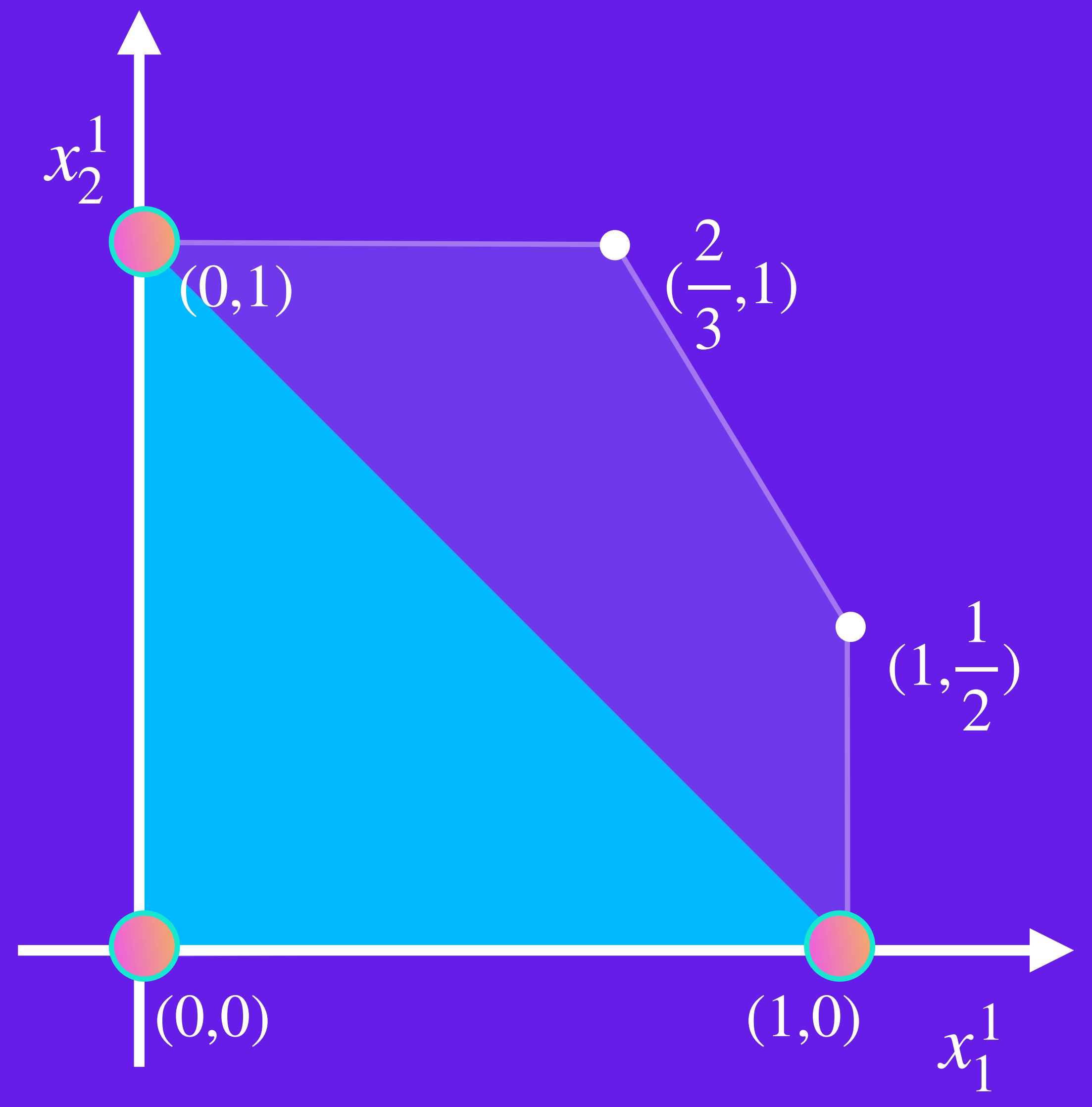
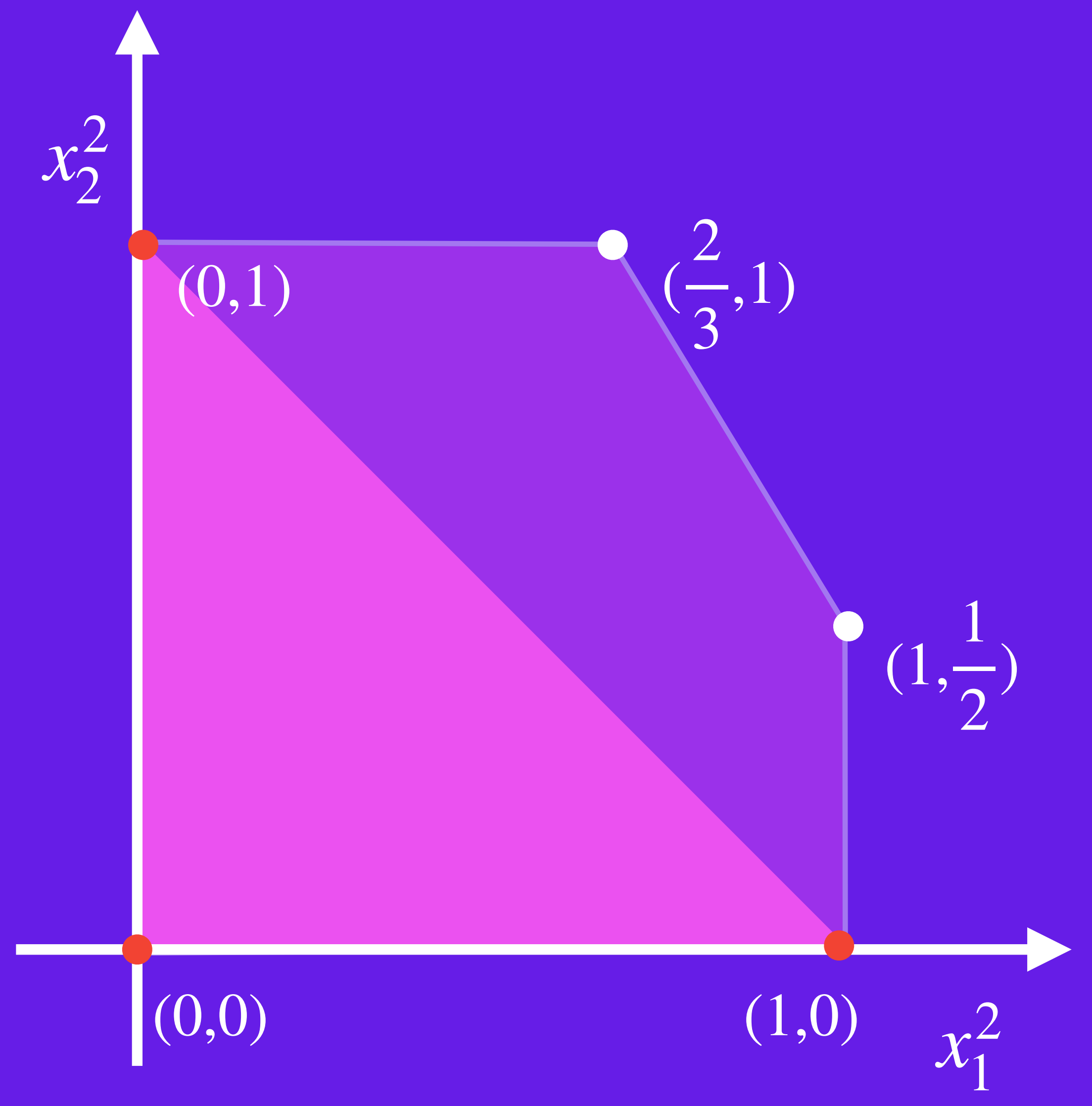
● Best-responses





$$\max_{x^1} \quad 6x_1^1 + x_2^1 \ -4x_1^1x_1^2 + 6x_2^1x_2^2$$

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$$x^1 \in \{0,1\}^2$$

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$$x^2 \in \{0,1\}^2$$

		x^2		
		(0,0)	(1,0)	(0,1)
x^1	(0,0)	0 0	0 4	0 2
	(1,0)	6 0	2 3	6 2
	(0,1)	1 0	1 2	7 1

$$\max_{x^2} \quad 4x_1^2 + 2x_2^2 \; - x_1^2 x_1^1 - x_2^2 x_2^1$$

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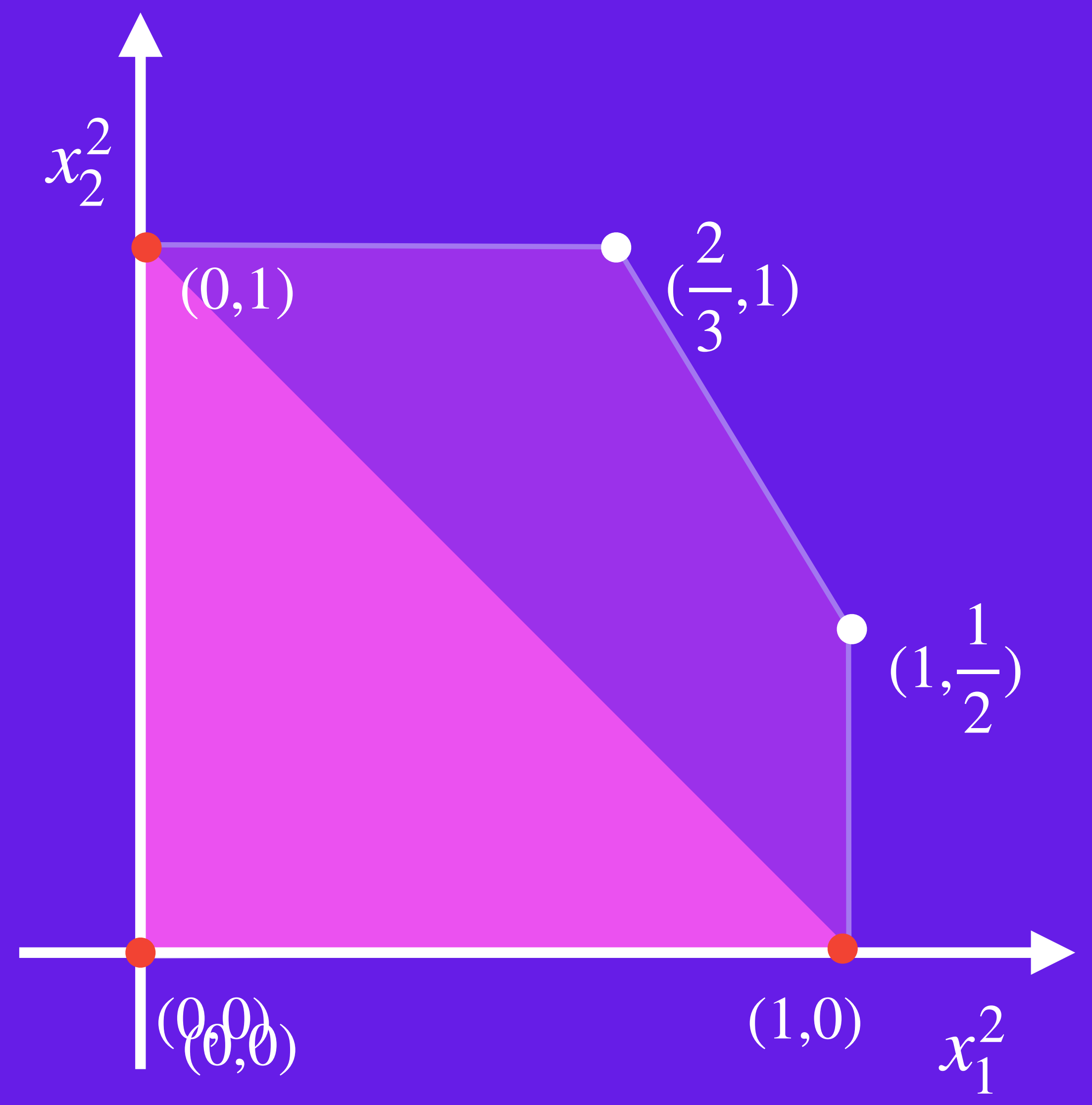

			x^2	
		(0,0)	(1,0)	(0,1)
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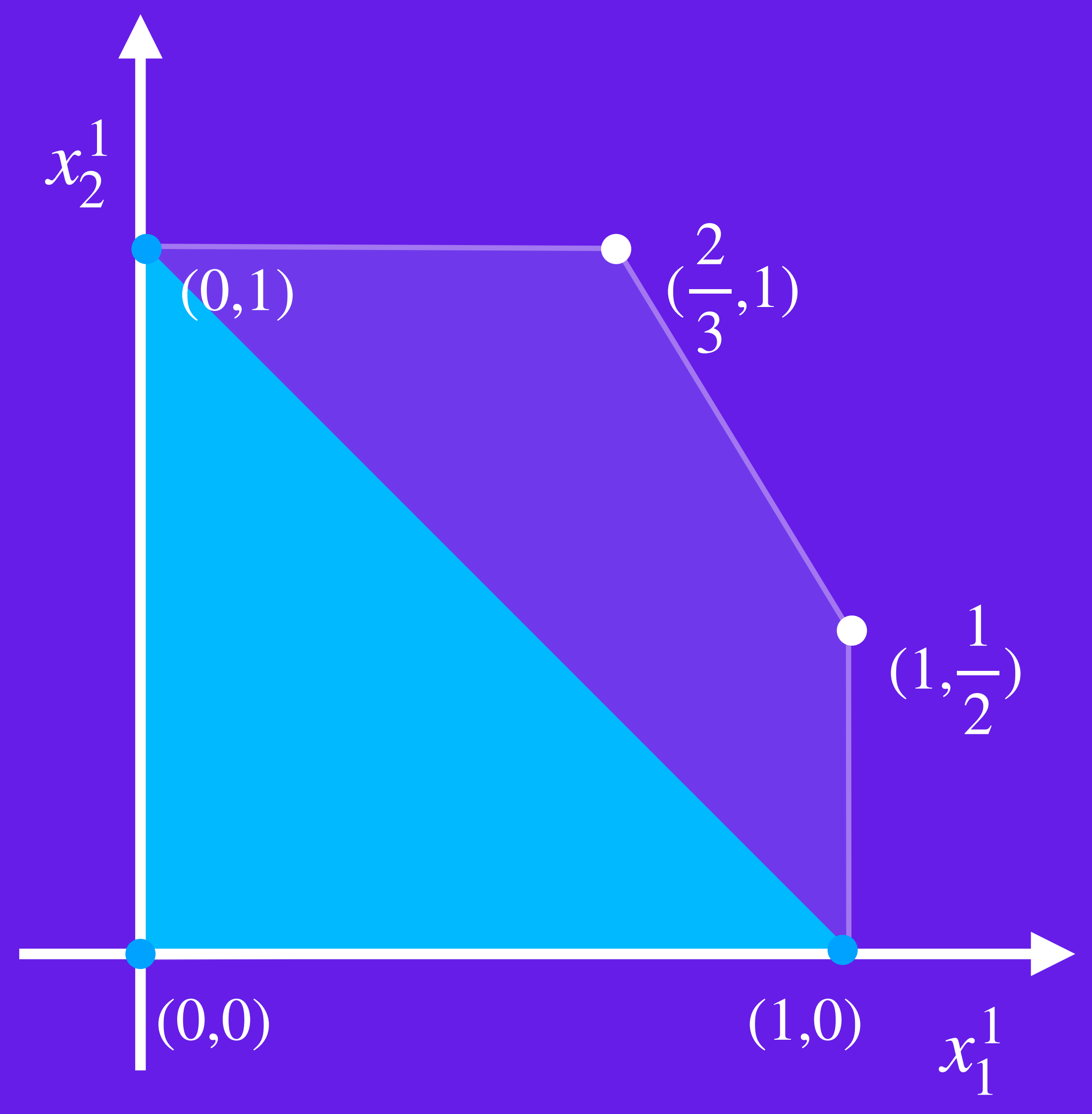

x^2

x^1	(0,0)	(1,0)	(0,1)
(0,0)	0 0	0 4	0 2
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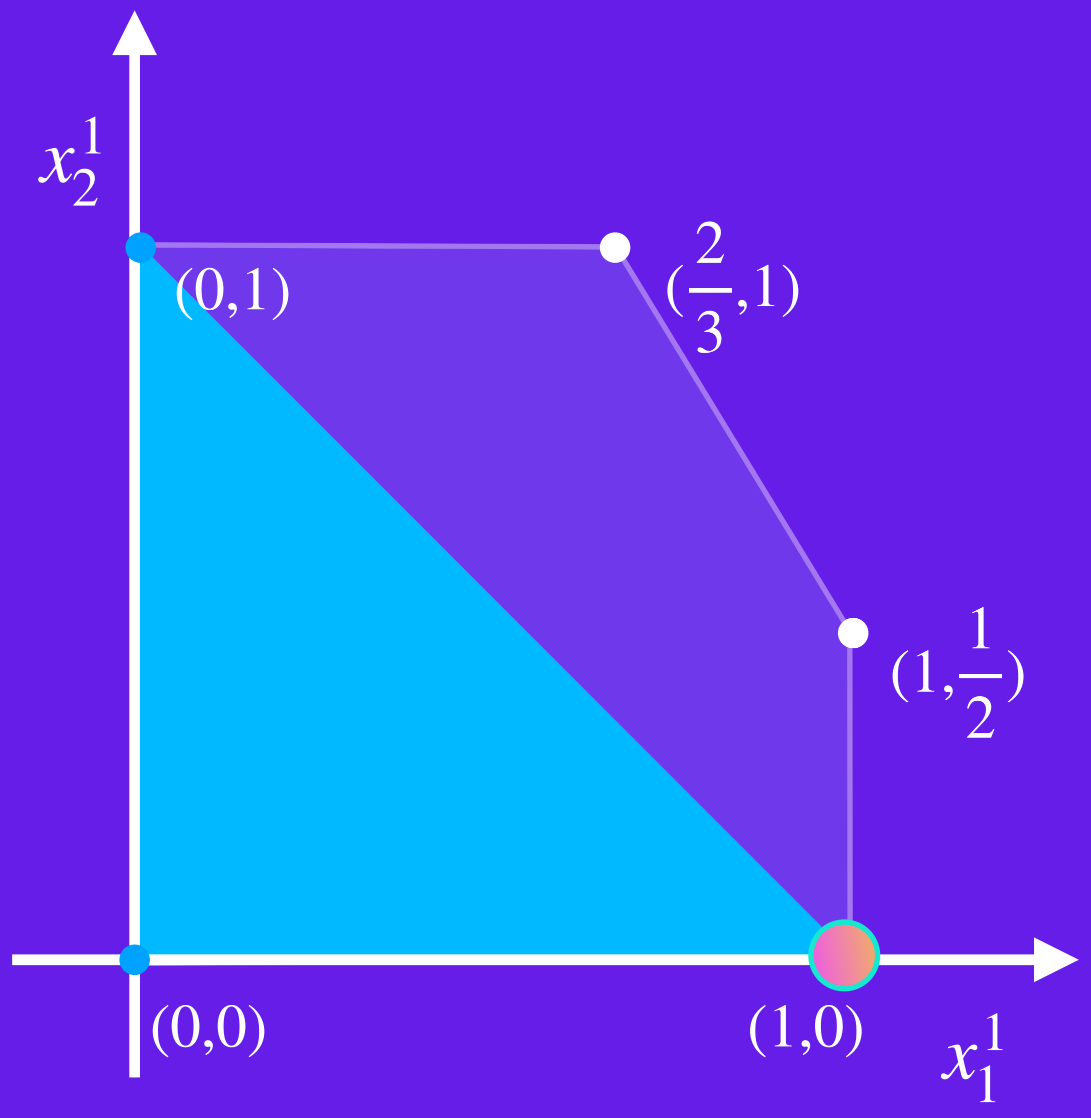

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