Mathematical Programming Games

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CANADA EXCELLENCE RESEARCH CHAIR
DATA SCIENCE FOR REAL-TIME DECISION-MAKING

POLYTECHNIQUE MONTRÉAL
TECHNOLOGICAL UNIVERSITY

Mathematical Programming

—

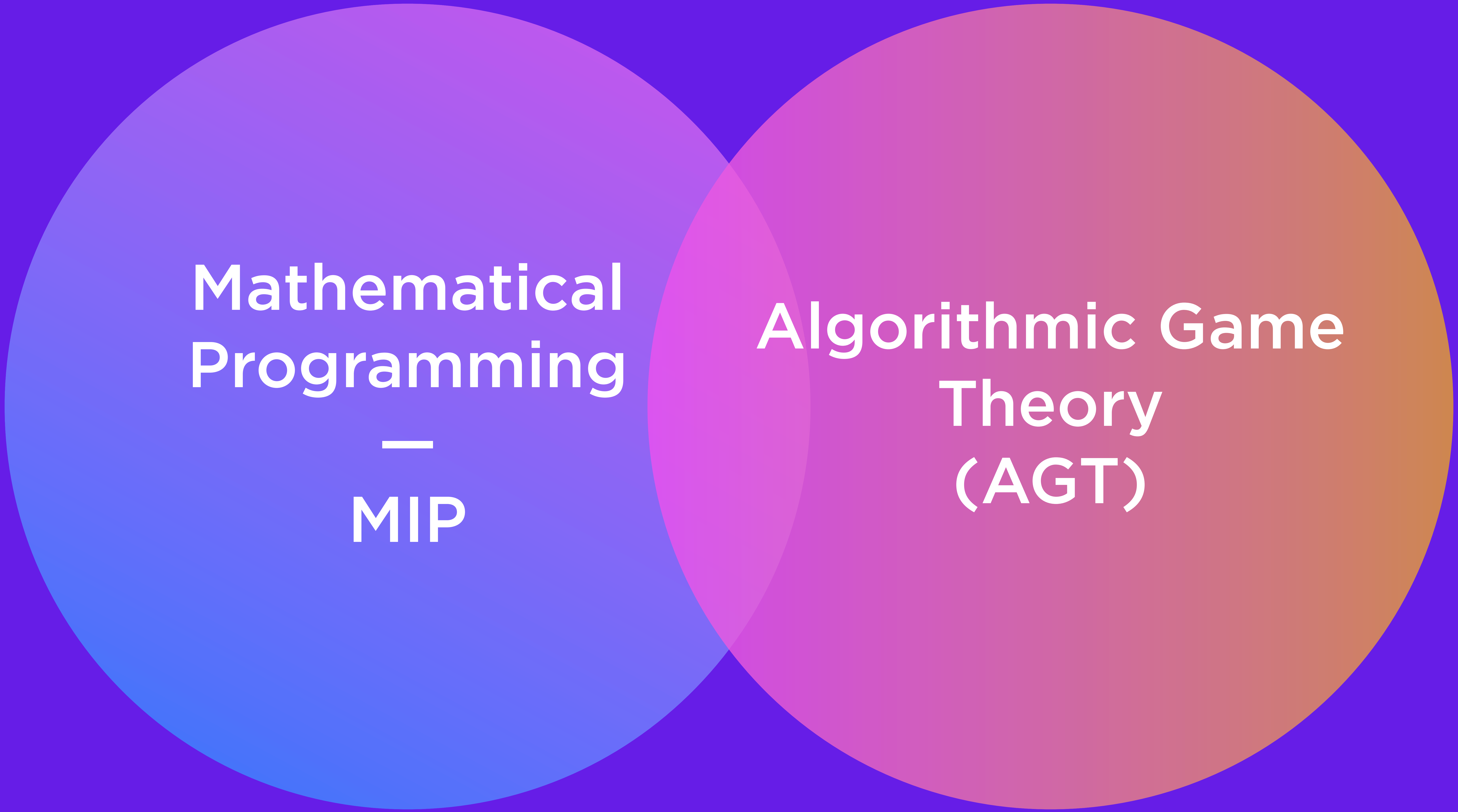
MIP

Mathematical Programming
—
MIP

- Powerful (and interpretable) modeling toolkit
- Vast algorithmic arsenal

Algorithmic Game Theory

- Interactions of algorithms and game theory
- Advanced modeling capabilities when **multiple agents interact**


Mathematical Programming
—
MIP
Algorithmic Game Theory
(AGT)

🧙🏽‍♂️ and 🧚🏼‍♂️

open two convenience stores in Magicville

$$\max_{x^1} \quad 6x_1^1 + x_2^1$$

$$3x_1^1 + 2x_2^1 \leq 4$$

$$x^1 \in \{0,1\}^2$$

Their items **interact**!

$$\max_{x^1} \quad 6x_1^1 + x_2^1 - 4x_1^1 x_1^2 + 3x_2^1 x_2^2$$

$$3x_1^1 + 2x_2^1 \leq 4$$

$$x^1 \in \{0,1\}^2$$

$$\max_{x^2} \quad 4x_1^2 + 2x_2^2 - x_1^2 x_1^1 - x_2^2 x_2^1$$

$$\text{s.t.} \quad 3x_1^2 + 2x_2^2 \leq 4$$

$$x^2 \in \{0,1\}^2$$

What problem are Wizard and Fairy solving?

How do we solve such problems, and what is a solution?

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They are solving what *we define* a ***Mathematical Programming Game*** (*MPG*), a ***simultaneous*** game among n players where each **rational** player $i = 1,2,\ldots,n$ solves the optimization problem

$$\max_{x^i} \{ \underline{f^i(x^i, x^{-i})} : x^i \in \underline{\mathcal{X}^i} \}$$

The payoff function for i

$$f^i(x^i, x^{-i}) : \prod_{j=1}^{n} \mathcal{X}^j \to \mathbb{R}$$

Is **parametrized in** x^{-i}

The set of actions for i

$\mathcal{X}^i$

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The payoff function for i

$$f^i(x^i, x^{-i}) : \prod_{j=1}^{n} \mathcal{X}^j \to \mathbb{R}$$

is **parametrized in** x^{-i}

The choices of i's opponents affect its payoff

The set of actions for i

$$\mathcal{X}^i$$

However, they do not affect its possible moves

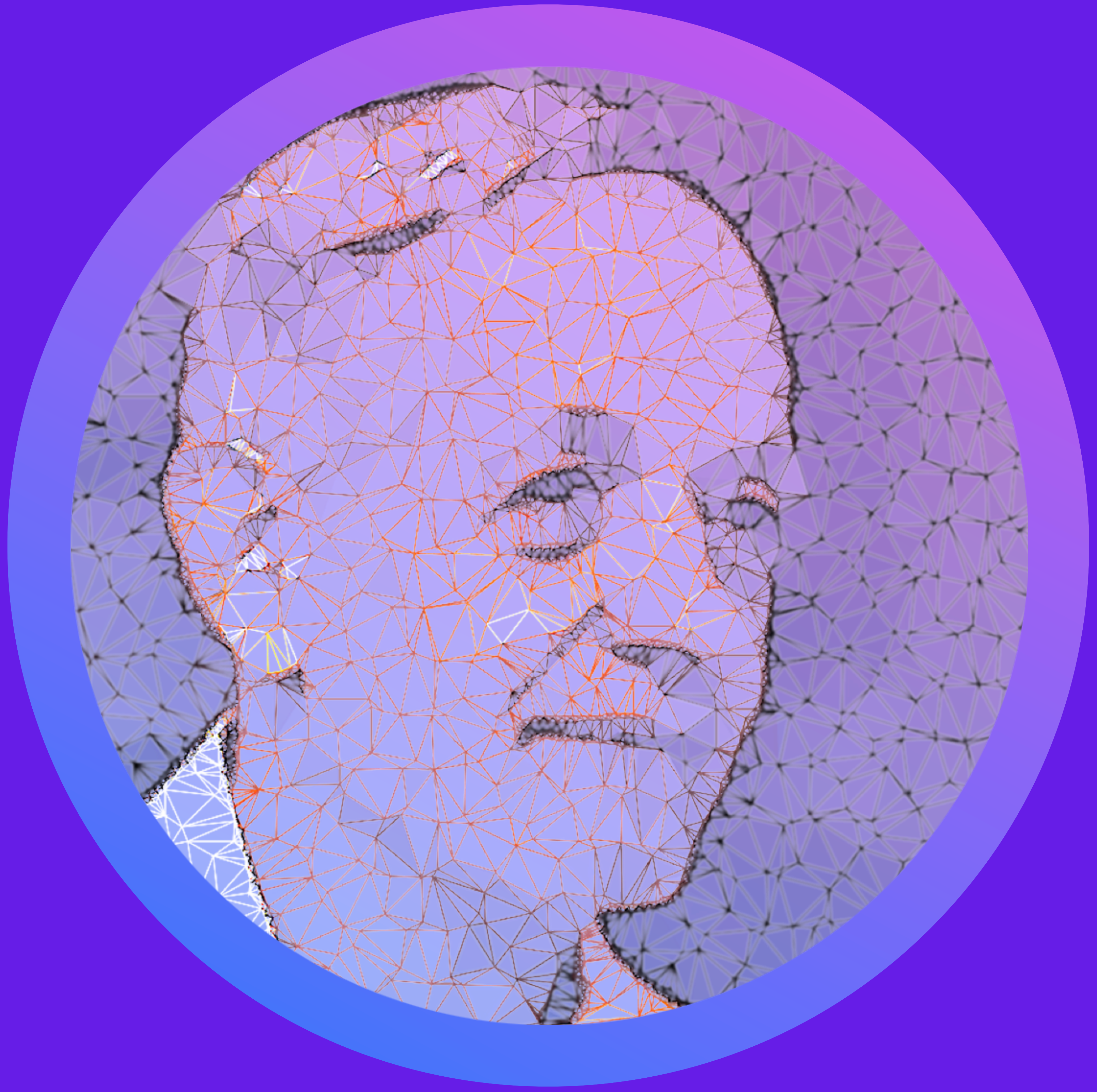
$$\max_{x^i} \{ f^i(x^i, x^{-i}) : x^i \in \mathcal{X}^i \}$$

Action Representation

Modeling Requirements

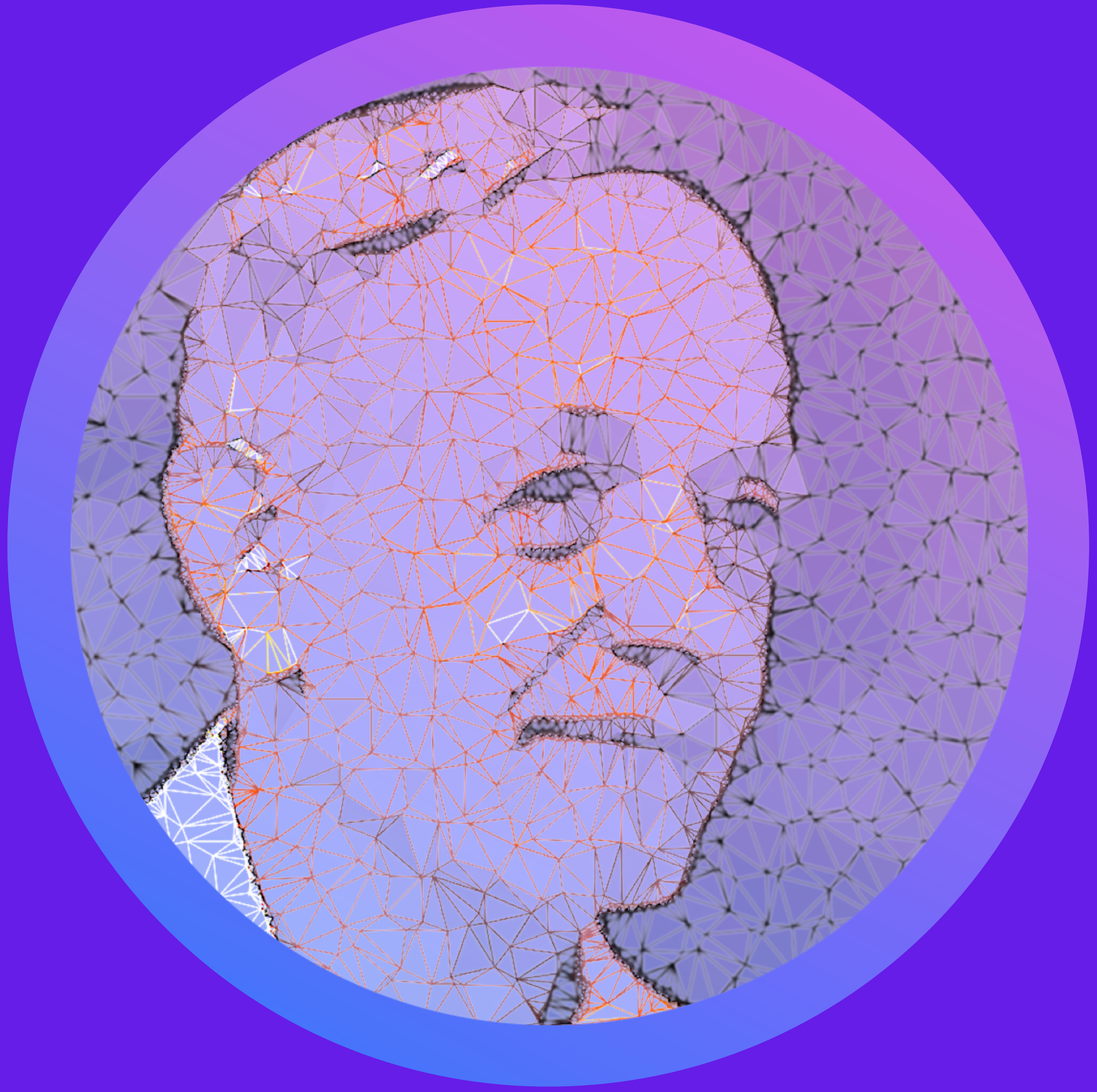
Language and Objectives

How do we solve such problems, and what is a solution?



In an *MPG*, *a vector* $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^n)$, *with* $\bar{x}^i \in \mathcal{X}^i$ *for any* $\boldsymbol{i}$, *is a Pure Nash Equilibrium* (**PNE**) *if*

$$f^i(\bar{x}^i, \bar{x}^{-i}) \geq f^i(\hat{x}^i, \bar{x}^{-i}) \quad \forall \hat{x}^i \in \mathcal{X}^i$$



Does at least **one exist**? **How hard** is to **compute** one?

How do we compute an NE, if any? And how do we **select one** if multiple exist?

How good — i.e., efficient — is this NE?

Objectives

Modeling

Can MPGs model real-world problems?

Efficiency

How do different NEs in MPG compare?

Algorithms

How do we compute and select NEs? What are the algorithms' properties?

Insights

Can NE promote socially beneficial outcomes?

Contributions

The Cut-And-Play

Computing Nash equilibria
via Outer Approximations

Algorithms

ZERO Regrets

Optimizing over Pure Nash equilibria via Integer Programming

Algorithms

Efficiency

When Nash Meets Stackelberg

Games among leaders of continuous Stackelberg games

Algorithms

Modeling

Insights

ZERO

A software library for Mathematical Programming Games

Algorithms

The Cut-and-Play Algorithm

Joint work with **Margarida Carvalho, Andrea Lodi** and **Sriram Sankaranarayanan**

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The Problem

RBGs

We consider *Reciprocally-Bilinear Games* (***RBGs***), namely MPGs where each player solves:

$$\max_{x^i} \{ f^i(x^i, x^{-i}) = (c^i)^\top x^i + (x^{-i})^\top C^i x^i : x^i \in \mathcal{X}^i \}$$

- The entries of c^i, C^i are integers,
- There is **common knowledge of rationality**, thus each player is **rational** and there is **complete information**,
- The game is **polyhedrally-representable** if $\operatorname{cl}\operatorname{conv}(\mathcal{X}^i)$ is a polyhedron for any i
- Blackbox to optimize a **linear function over** $\mathcal{X}^i$

Contributions

Algorithms

Cutting plane algorithm: computes **(Mixed) Nash equilibria** (M*NEs*) + An enhanced Separation Oracle 🔮

The first algorithm to work with iteratively refined **outer approximations** of player's feasible sets (convex hulls)

Integrates **integer programming machinery**

Practical

Extensive testing on Knapsack Games and games among bilevel leaders (NASPs)

Lemke-Howson and Complementarity Methods

Lemke and Howson, 1964;
Rosenmüller, 1971;
Wilson, 1971;
Avis et al., 2010;
Audet et al., 2006;
Sagratella, 2016.

Homotopy-based

Scarf, 1967.

Equilibrium Programming

Facchinei and Pang, 2003;
Sagratella, 2016.

Support Enumeration

Sandholm et al., 2005;
Porter et al., 2008;

MIP

Sandholm et al., 2005;
Cronert and Minner, 2021.

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MIP

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Complementarities

LCPs

A ***linear complementarity problem*** (***LCP***) is the problem of finding a vector x such that:

$$z = Mx + q,\ x^\top z = 0$$
$$x \geq 0,\ z \geq 0$$

or show that no such vector exist.

$$\max_{x^1} \quad 6x_1^1 + x_2^1 - 4x_1^1x_1^2 + 3x_2^1x_2^2$$

$$\text{s.t.} \quad 3x_1^1 + 2x_2^1 \leq 4$$

$$x^1 \in \{0,1\}^2$$

$$\max_{x^1} \quad (c^1)^\top x^1 + (x^{-1})^\top C^1 x^1$$
$$\text{s.t.} \quad A^1 x^1 \leq b^1$$
$$x^1 \in \{0,1\}^m$$

$$\max_{x^2} \quad (c^2)^\top x^2 + (x^{-2})^\top C^2 x^2$$
$$\text{s.t.} \quad A^2 x^2 \leq b^2$$
$$x^2 \in \{0,1\}^m$$

$$q = \begin{bmatrix} c^1 \\ b^1 \\ \vdots \\ c^n \\ b^n \end{bmatrix} \qquad M = \begin{bmatrix} C^1 x^{-1} & A^{1\top} \\ -A^1 & 0 \\ \vdots & \\ C^n x^{-n} & A^{n\top} \\ -A^n & 0 \end{bmatrix}$$

$$z = M\sigma + q, \ \sigma^\top z = 0$$
$$\sigma \geq 0, \ z \geq 0$$

Provides all the MNEs for the game?

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$$\sigma \geq 0,\ z \geq 0$$

Provides all the MNEs for the game?

Yes

If A^i, b^i **describe** cl conv$(\mathscr{X}^i)$

- May be prohibitive in practice

Maybe

If A^i, b^i **do not** describe cl conv$(\mathscr{X}^i)$

- Some MNEs may be excluded
- Some spurious MNEs may be introduced
- May not give bounds, as in Optimization

$$0 \leq \sigma \perp z = (M_t\sigma + q_t) \geq 0$$

Provides all the MNEs for the game?

Yes

If A^i, b^i **describe** cl conv$(\mathcal{X}^i)$

THEOREM *(the shortened version)*

Given an RBG $\mathbf{G}$ and a copy of it $\tilde{\mathbf{G}}$ where the feasible region of player $\boldsymbol{i}$ is cl conv $(\mathcal{X}^i)$ (instead of $\mathcal{X}^i$), then:

- *For any PNE $\tilde{\boldsymbol{\sigma}}$ of $\tilde{\mathbf{G}}$, there exists an MNE $\hat{\boldsymbol{\sigma}}$ of $\mathbf{G}$ so that each player get the same payoff in $\tilde{\mathbf{G}}$ and $\mathbf{G}$*
- *If $\tilde{\mathbf{G}}$ has no PNEs, then $\mathbf{G}$ has no MNEs.*

The Idea

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At each iteration, either we **find an MNE** for G or we **refine the approximation** in $\tilde{G}$

Approximation

PAG

Given the polyhedrally-representable *RBG* $\boldsymbol{G}$, we construct ***polyhedral approximate game*** $\tilde{G}$ where each $\boldsymbol{i}$ solves instead:

$$\max_{x^i}\{f^i(x^i, x^{-i}) = (c^i)^\top x^i + (x^{-i})^\top C^i x^i : x^i \in \tilde{\mathcal{X}}^i\}$$

$$\tilde{\mathcal{X}}^i := \{\tilde{A}^i x^i \leq \tilde{b}^i, x^i \geq 0\}, \mathcal{X}^i \subseteq \text{cl conv}(\mathcal{X}^i) \subseteq \tilde{\mathcal{X}}^i$$

Namely, $\tilde{\mathcal{X}}^i$ **outer approximates cl conv**$(\mathcal{X}^i)$

Finding MNEs

The LCP

$$\max_{x^i}\{f^i(x^i, x^{-i}) = (c^i)^\top x^i + (x^{-i})^\top C^i x^i : x^i \in \tilde{\mathcal{X}}^i\}$$

$$\tilde{\mathcal{X}}^i := \{\tilde{A}^i x^i \le \tilde{b}^i, x^i \ge 0\}, \mathcal{X}^i \subseteq \text{cl conv}(\mathcal{X}^i) \subseteq \tilde{\mathcal{X}}^i$$

$$\tilde{q} = \begin{bmatrix} c^1 \\ \tilde{b}^1 \\ \vdots \\ c^n \\ \tilde{b}^n \end{bmatrix} \quad \tilde{M} = \begin{bmatrix} C^1 x^{-1} & \tilde{A}^{1\top} \\ -\tilde{A}^1 & 0 \\ \vdots & \\ C^n x^{-n} & \tilde{A}^{n\top} \\ -\tilde{A}^n & 0 \end{bmatrix}$$

$$\tilde{z} = \tilde{M}\tilde{\sigma} + \tilde{q},\ \tilde{\sigma}^\top \tilde{z} = 0$$

$$\tilde{\sigma} \ge 0,\ \tilde{z} \ge 0$$

Is $\tilde{\sigma}$ an MNE for G ?

Ask the Oracle

Oracle

Enhanced Sep. Oracle

Given a point $\bar{x}\ (=\tilde{\sigma}^i)$ *and* $\mathscr{X}\ (=\mathscr{X}^i)$, *the* **Enhanced Separation Oracle (ESO)** determines that either

$\bar{x} \in$ cl conv $(\mathscr{X})$ and an "extended proof"

At least one $\bar{x} \notin$ cl conv $(\mathscr{X})$ + a cut for cl conv$(\mathscr{X})$ and $\bar{x}$

The ***extended proof*** is the support of $\bar{x}$, i.e. convex combination of elements in ext(cl conv$(\mathscr{X})$) and conic of rec(cl conv$(\mathscr{X})$).

In practice, the oracle builds a $\mathscr{V}$-polyhedral inner-approximation of cl conv $(\mathscr{X})$

Enhanced Separation Oracle

INPUT: A point $\bar{x}\ (=\tilde{\sigma}^i)$ and $\mathscr{X}\ (=\mathscr{X}^i)$ (a tolerance ε)

OUTPUT: yes and proof or no and a cut

$V = R = \varnothing$ or storage

```
Repeat:
  W ← conv(V) + cone(R)            👀 Inner approximation of cl conv(X)
  If x̄ ∈ W: return yes and proof of inclusion
  If x̄ ∉ W:
      π̄ᵀx ≤ π̄₀ separates x̄ and W
      G ← max_x{π̄ᵀx : x ∈ X} with ν maximizer
      If G = ∞: R ← R ∪ {r} with r extreme ray
      Else:
         If π̄ᵀν < π̄ᵀx̄: return no and π̄ᵀx ≤ π̄ᵀν
         Else: V ← V ∪ {ν}
```

Repeat:

- $\mathscr{W} \leftarrow \mathrm{conv}(V) + \mathrm{cone}(R)$ 👀 *Inner approximation of cl conv$(\mathscr{X})$*
- If $\bar{x} \in \mathscr{W}$: return yes and proof of inclusion
- If $\bar{x} \notin \mathscr{W}$:
  - $\bar{\pi}^\top x \leq \bar{\pi}_0$ separates $\bar{x}$ and $\mathscr{W}$
  - $\mathscr{G} \leftarrow \max_x\{\bar{\pi}^\top x : x \in \mathscr{X}\}$ with ν maximizer
  - If $\mathscr{G} = \infty$: $R \leftarrow R \cup \{r\}$ with r extreme ray
  - Else:
    - If $\bar{\pi}^\top \nu < \bar{\pi}^\top \bar{x}$: return no and $\bar{\pi}^\top x \leq \bar{\pi}^\top \nu$
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I'm an LP

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Enhanced Separation Oracle

$$\max_{\pi,\pi_0} \quad \bar{x}^\top \pi - \pi_0$$

$$\pi v_k^\top - \pi_0 \leq 0 \quad \forall v_k \in V$$

$$\pi r_j^\top \leq 0 \quad \forall r_j \in R$$

$$e^\top (u + v) = 1$$

$$\pi + u - v = 0$$

$$u, v \geq 0$$

YES

Objective is 0

Enhanced Separation Oracle

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The Cut-and-Play

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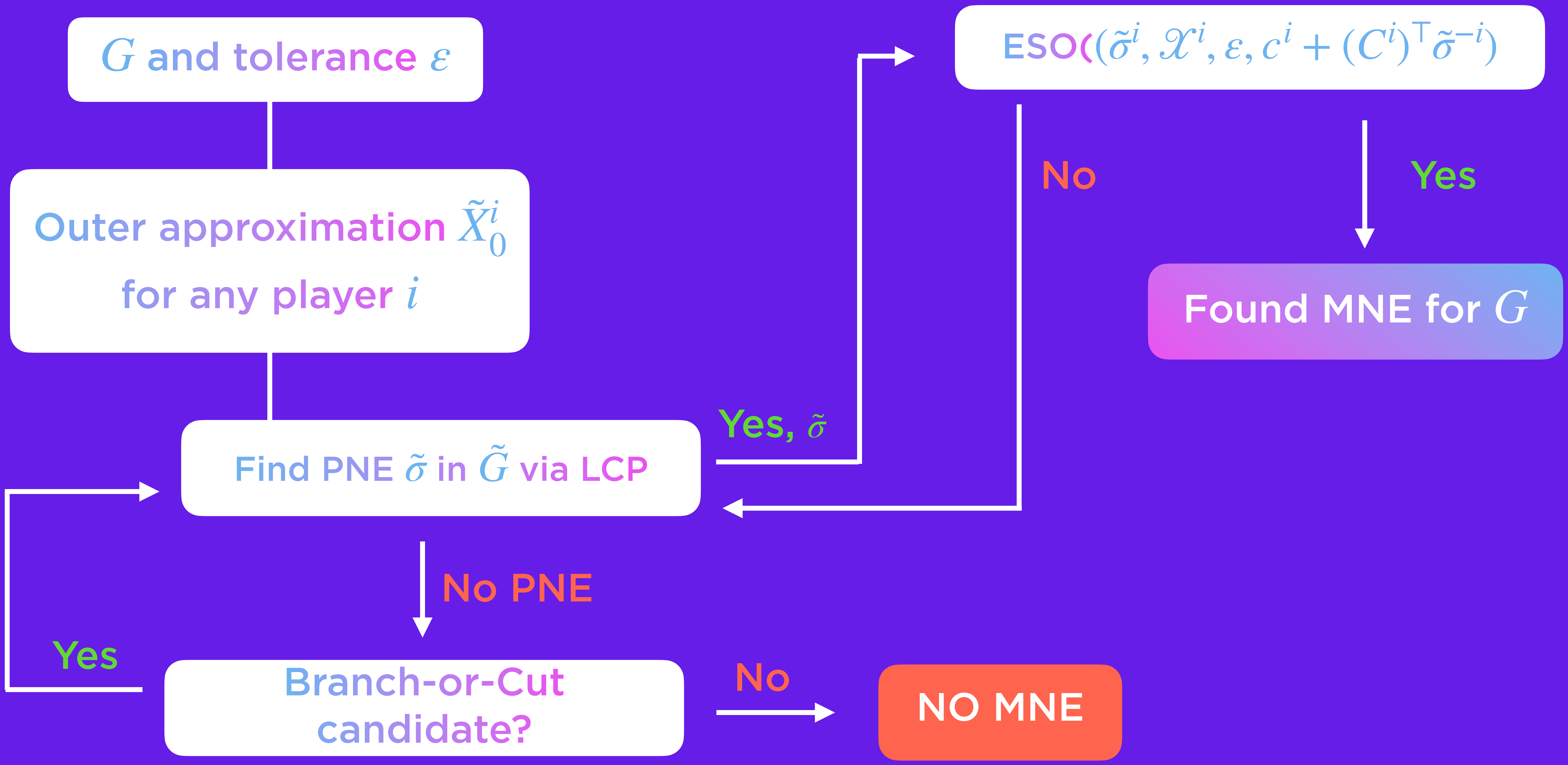

The Cut-And-Play

For any i, any inequality valid for cl conv$(\mathscr{X}^i)$ works for the algorithm (at any step).

Further, a MIP solver can handle the LCP

Experiments

Knapsack Game (*KPG*)

As for Wizard and Fairy, each player solves a binary Knapsack problem with some interaction terms in the objective

$$\max_{x^i} \left\{ \sum_{j=1}^{m} p_j^i x_j^i + \sum_{k=1, k \neq i}^{n} \sum_{j=1}^{m} C_{k,j}^i x_j^i x_j^k : \sum_{j=1}^{m} w_j^i x_j^i \leq b^i, \mathbf{x}^i \in \{0,1\}^m \right\}$$

W.l.o.g., each player controls m items

Knapsack Game

Algo	Obj	A	Geo t (s)	#TL	#It	Cuts	MIP	Efficiency (“~PoS”)
m-SGM	-		0.73	0	8.43	-	-	1.37
CnP-MIP	SocialW	-1	6.58	0	7.80	9.57	0.00	1.21
	SocialW	0	6.13	0	5.73	6.47	2.30	1.22
	SocialW	1	6.31	0	3.50	9.6	7.47	1.21
CnP-PATH	-	-1	0.36	0	7.60	10.2	0.00	1.21
	-	0	0.05	0	5.27	5.9	2.07	1.35
	-	1	0.04	0	3.23	8.87	7.10	1.33
m-SGM	-	-1	20.86	6	18.58	-	-	1.50
CnP-MIP	SocialW	0	61.08	0	13.70	17.0	0.00	1.23
	SocialW	1	57.85	1	11.62	12.62	3.45	1.26
	SocialW	-1	68.20	0	9.48	16.8	10.32	1.23
CnP-PATH	-	0	6.68	0	13.55	16.35	0.00	1.24
	-	1	4.48	0	9.62	10.25	2.42	1.30
	-	-1	4.32	0	8.22	14.35	8.43	1.30

Knapsack Game

Small $nm \leq 80$

Large $nm > 80$

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CnP-PATH	-	-1	0.36	0	7.60	10.2	0.00	1.21
	-	0	0.05	0	5.27	5.9	2.07	1.35
	-	1	0.04	0	3.23	8.87	7.10	1.33
m-SGM	-	-1	20.86	6	18.58	-	-	1.50
CnP-MIP	SocialW	0	61.08	0	13.70	17.0	0.00	1.23
	SocialW	1	57.85	1	11.62	12.62	3.45	1.26
	SocialW	-1	68.20	0	9.48	16.8	10.32	1.23
CnP-PATH	-	0	6.68	0	13.55	16.35	0.00	1.24
	-	1	4.48	0	9.62	10.25	2.42	1.30
	-	-1	4.32	0	8.22	14.35	8.43	1.30

Knapsack Game

Algo	Obj	A	Geo t (s)	#TL	#It	Cuts	MIP	Efficiency ("~PoS")
m-SGM	-		0.73	0	8.43	-	-	1.37
CnP-MIP	SocialW	-1	6.58	0	7.80	9.57	0.00	1.21
	SocialW	0	6.13	0	5.73	6.47	2.30	1.22
	SocialW	1	6.31	0	3.50	9.6	7.47	1.21
CnP-PATH	-	-1	0.36	0	7.60	10.2	0.00	1.21
	-	0	0.05	0	5.27	5.9	2.07	1.35
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Knapsack Game

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m-SGM	-	-	0.73	0	8.43	-	-	1.37
CnP-MIP	SocialW	❌	6.58	0	7.80	9.57	0.00	1.21
	SocialW	✅	6.13	0	5.73	6.47	2.30	1.22
	SocialW	✅✅	6.31	0	3.50	9.6	7.47	1.21
CnP-PATH	-	❌	0.36	0	7.60	10.2	0.00	1.21
	-	✅	0.05	0	5.27	5.9	2.07	1.35
	-	✅✅	0.04	0	3.23	8.87	7.10	1.33
m-SGM	-	-	20.86	6	18.58	-	-	1.50
CnP-MIP	SocialW	❌	61.08	0	13.70	17.0	0.00	1.23
	SocialW	✅	57.85	1	11.62	12.62	3.45	1.26
	SocialW	✅✅	68.20	0	9.48	16.8	10.32	1.23
CnP-PATH	-	❌	6.68	0	13.55	16.35	0.00	1.24
	-	✅	4.48	0	9.62	10.25	2.42	1.30
	-	✅✅	4.32	0	8.22	14.35	8.43	1.30

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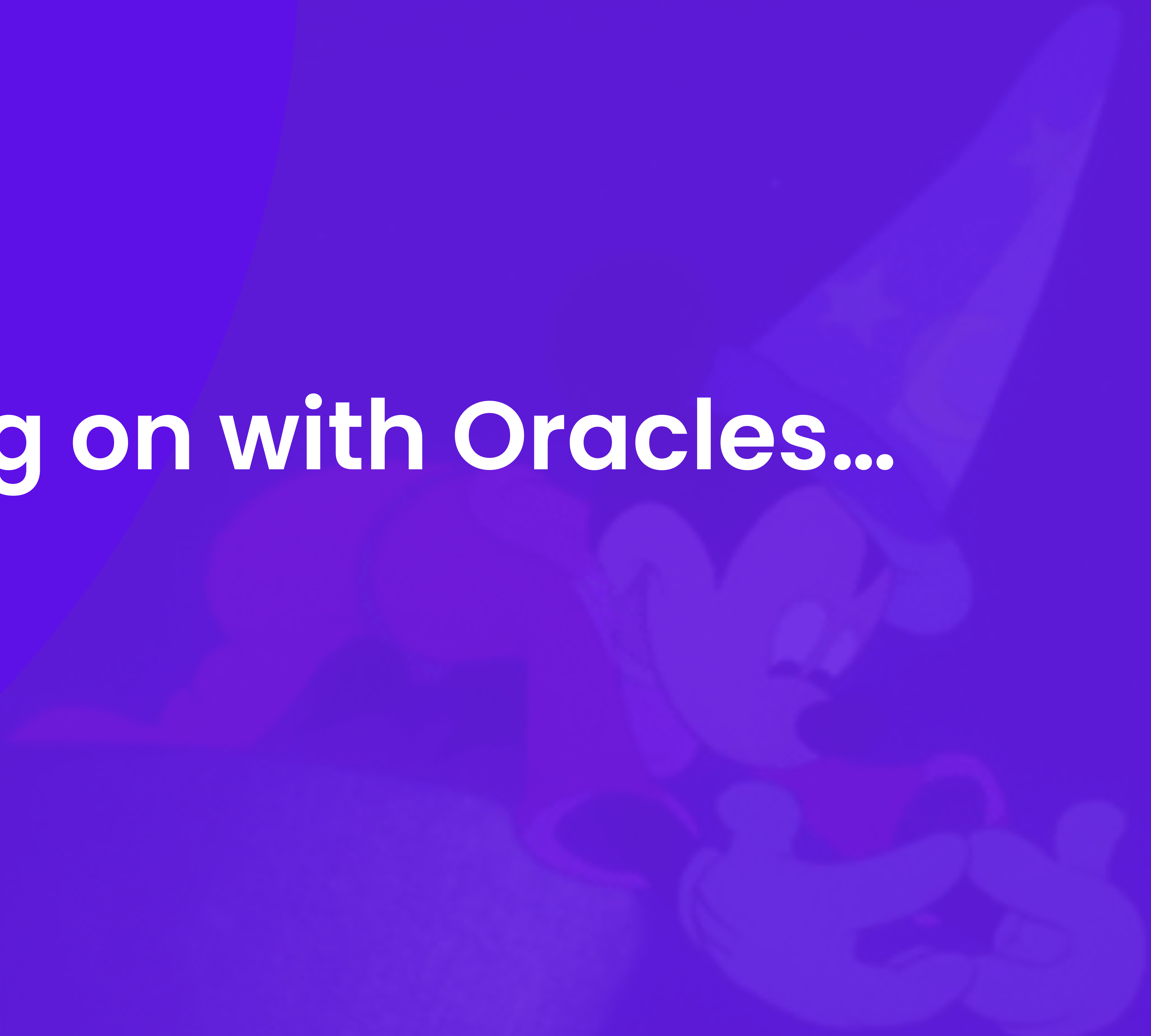
Keeping on with Oracles...



The ZERO Regrets Algorithm

Joint work with **Rosario Scatamacchia**

2

Their items **interact**!

$$\max_{x^1} \quad 6x_1^1 + x_2^1 - 4x_1^1 x_1^2 + 3x_2^1 x_2^2$$

$$\text{s.t.} \quad 3x_1^1 + 2x_2^1 \leq 4$$

$$x^1 \in \{0,1\}^2$$

$$\max_{x^2} \quad 4x_1^2 + 2x_2^2 - x_1^2 x_1^1 - x_2^2 x_2^1$$

$$\text{s.t.} \quad 3x_1^2 + 2x_2^2 \leq 4$$

$$x^2 \in \{0,1\}^2$$

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$$\text{s.t.} \quad 3x_1^2 + 2x_2^2 \leq 4$$
$$x^2 \in \{0,1\}^2$$

How good is a NE?

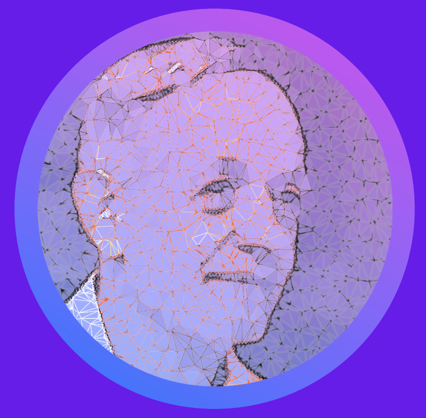
Could we select PNEs?

How good is a NE?

How good is a NE?

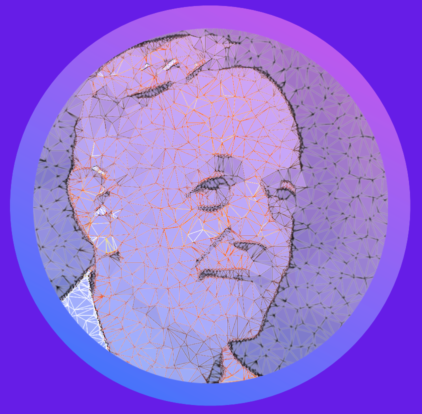
$$\max_{x^1} \quad 6x_1^1 + x_2^1 - 4x_1^1 x_1^2 + 3x_2^1 x_2^2$$
$$\text{s.t.} \quad 3x_1^1 + 2x_2^1 \leq 4$$
$$x^1 \in \{0,1\}^2$$

$$\max_{x^2} \quad 4x_1^2 + 2x_2^2 - x_1^2 x_1^1 - x_2^2 x_2^1$$
$$\text{s.t.} \quad 3x_1^2 + 2x_2^2 \leq 4$$
$$x^2 \in \{0,1\}^2$$

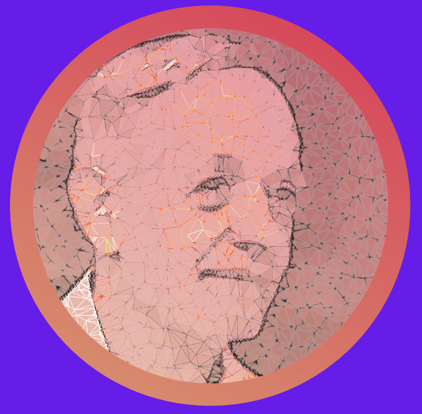


$(\bar{x}_1^1, \bar{x}_2^1) = (1,0)$ and $(\bar{x}_1^2, \bar{x}_2^2) = (1,0)$ with $W = 2 + 3 = 5$

$(\bar{x}_1^1, \bar{x}_2^1) = (1,0)$ and $(\bar{x}_1^2, \bar{x}_2^2) = (0,1)$ $W = 6 + 2 = 8$



$$= \frac{\text{Optimal Social Welfare}}{\text{"Best" NE}} = PoS$$



$$= \frac{\text{Optimal Social Welfare}}{\text{"Worst" NE}} = PoA$$

The Problem

IPGs

(Köppe et al., 2011)

We consider *Integer Programming Games* (***IPGs***), namely MPGs where each player solves an Integer Program. Specifically, we consider IPGs so that each player $i = 1,2,\ldots,n$ solves

$$\max_{x^i}\{u^i(x^i, x^{-i}) : x^i \in \mathcal{X}^i\},\ \mathcal{X}^i := \{A^i x^i \le b^i, x^i \in \mathbb{Z}^m\}$$

- The entries of A^i, b^i and the coefficients of $u^i(x^i, x^{-i})$ are integers,
- There is **common knowledge of rationality**, thus each player is **rational** and there is **complete information**,
- The payoffs u^i can be **linearized** in the space of all players variables

Literature

IPGs

Introduced by ***Köppe et al., 2011***

Carvalho et al., 2018 prove it is Σ_2^p-hard to determine if a NE exists, in general

Carvalho et al., 2020 provide also an algorithm to compute their NE through a smart support exploration

Cronert and Minner, 2020 provide an approach to enumerate NE, yet, it does not scale in practice

Contributions

Theoretical

Polyhedral characterization: inequalities, polyhedral closures

Algorithms

Cutting plane algorithm: computes, ***selects***, enumerates **Pure Nash equilibria** (*PNEs*). A Separation Oracle🔮 and where to find it

Practical

Knapsack Game complexity and tests, and **Network Formation Game** PNE selection

Higher Dimensions

Lifted Space

The sets $\mathscr{K}$ and $\mathcal{E}$

Let $\mathscr{N} := \{x = (x^1, \ldots, x^n) : x$ is a PNE$\}$. Consider the **lifted space** $\mathscr{K}$, where z is induced by the linearization and the choice of x:

$$\mathscr{K} = \{(x^1, \ldots, x^n, z) \in \mathscr{L}, x^i \in \mathscr{X}^i \text{ for any } i = 1, \ldots, n\}$$

The $\text{proj}_x(\text{conv}(\mathscr{K}))$ encompasses **all the strategy profiles**.

The polyhedron $\mathcal{E}$:

$$\mathcal{E} = \{(x^1, \ldots, x^n, z) \in \text{conv}(\mathscr{K}) : (x^1, \ldots, x^n) \in \text{conv}(\mathscr{N})\}$$

Is the so-called **Perfect Equilibrium Formulation**

Lifted Space

The sets $\mathscr{K}$ and $\mathcal{E}$

$$\mathcal{E} = \{(x^1, \ldots, x^n, z) \in \text{conv}(\mathscr{K}) : (x^1, \ldots, x^n) \in \text{conv}(\mathcal{N})\}$$

Is the so-called **Perfect Equilibrium Formulation**

Namely, optimizing a function $S(x, z) : \mathscr{K} \to \mathbb{R}$ over it gives the **PNE maximizing** S for any **vertex of** ε

The Idea

The Idea

Start from conv($\mathscr{K}$) and get to some *intermediate polyhedron* over which optimizing $S(x, z)$ yields a point $(\bar{x}, \bar{z}) \in \mathcal{E}$ with $\bar{x} \in \mathcal{N}$

Inequalities

Equilibrium Inequality

An inequality is an **equilibrium inequality** *if it is valid for* $\mathcal{E}$

Consider now a player i and one of its strategies $\tilde{x}^i \in \mathcal{X}^i$:

$$u^i(\tilde{x}^i, x^{-i}) \leq u^i(x^i, x^{-i})$$

is necessarily an **equilibrium inequality.** Further, it is **linear** in the space of $\mathcal{K}$

Are these inequalities enough for $\mathcal{E}$?

Closure

Equilibrium Closure

THEOREM

The equilibrium closure of conv$(\mathscr{K})$ given by the set of equilibrium inequalities from before is given by:

$$P^e := \left\{ (x,z) \in \mathrm{conv}(\mathcal{K}) \;\middle|\; \begin{array}{l} u^i(\tilde{x}^i, x^{-i}) \leq u^i(x^i, x^{-i}) \\ \forall \tilde{x} : \tilde{x}^i \in \mathcal{BR}(i, \tilde{x}^{-i}), i = 1, \ldots, n \end{array} \right\}$$

where $\mathscr{BR}(i, \tilde{x}^{-i})$ are the best-responses of i given $\tilde{x}^{-i}$. Then:

P^e is a rational polyhedron, and

int(P^e) contains no points $(\bar{x}, \bar{z}) : \bar{x} \in \mathbb{Z}^{mn}$, and

$P^e = \varepsilon$.

An Oracle

Oracle

Equilibrium Oracle

Given a point $(\bar{x}, \bar{z})$ *and* $\mathcal{E}$*, the* **equilibrium separation problem** is the task of determining that either:

$(\bar{x}, \bar{z}) \in \mathcal{E}$ **and** 🏝️

$(\bar{x}, \bar{z}) \notin \mathcal{E}$ **+ an equilibrium inequality**

🔮

The Equilibrium Separation Oracle

Equilibrium Separation Oracle

INPUT: A profile $(\bar{x}, \bar{z})$ and an IPG Instance

OUTPUT: yes or no and Φ

For every player $i = 1,2,\ldots,n$

$\hat{x}^i \leftarrow \max_{x^i}\{u^i(x^i, \bar{x}^{-i}) : A^i x^i \leq b^i, x^i \in \mathbb{Z}^m\}$

If $u^i(\hat{x}^i, \bar{x}^{-i}) > u^i(\bar{x}^i, \bar{x}^{-i})$:

Add $u^i(\hat{x}^i, x^{-i}) \leq u^i(x^i, x^{-i})$ to Φ

If Φ is empty: return yes

Else: return no and Φ

ZERO Regrets

ZERO Regrets

INPUT: An IPG Instance and f

OUTPUT: A PNE $\bar{x}$

Set $\Phi = \{0 \leq 1\}$

While (STOP)

$$(\bar{x}, \bar{z}) = \arg \max_{x^1, \ldots, x^n, z} \{f(x,z) : (x,z) \in \mathrm{conv}(\mathscr{K}), \quad \Phi\}$$

If 🔮$(\bar{x}, \bar{z})$ says yes: $\bar{x}$ is the PNE maximizing f

Else 🔮$(\bar{x}, \bar{z})$ says no:

add at least a violated equilibrium inequality to Φ

ZERO Regrets

INPUT: An IPG Instance and f

OUTPUT: A PNE $\bar{x}$

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Set Φ = {0 ≤ 1}
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$$(\bar{x}, \bar{z}) = \arg\max_{x^1,\dots,x^n,z} \{ f(x,z) : (x,z) \in \mathrm{conv}(\mathcal{K}), \quad \Phi \}$$

👀 *I'm a MIP*

```
  If 🔮(x̄, z̄) says yes: x̄ is the PNE maximizing f
  Else 🔮(x̄, z̄) says no:
    add at least a violated equilibrium
    inequality to Φ
```

ZERO Regrets

INPUT: An IPG Instance and f

OUTPUT: A PNE $\bar{x}$

Set $\Phi = \{0 \leq 1\}$

While (STOP)

$$(\bar{x}, \bar{z}) = \arg \max_{x^1,\dots,x^n,z} \{f(x,z) : (x,z) \in \text{conv}(\mathcal{K}), \quad \Phi\}$$

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add at least a violated equilibrium inequality to Φ

Experiments

Knapsack Game (*KPG*)

THEOREM

Deciding if a Knapsack Game instance has a *PNE is* Σ_2^p-complete.

PROPOSITION

In a KPG, the PoA and the PoS may be arbitrarily bad (for PNEs).

Knapsack Game

(n, m, t)	EQIne	Time (s)	PoS	#TL
(2, 25, A)	10.67	0.08	1.04	0
(2, 25, B)	15.67	0.17	1.02	0
(2, 25, C)	40.00	1.52	1.06	0
(2, 50, A)	15.00	0.22	1.02	0
(2, 50, B)	41.67	1.27	1.01	0
(2, 50, C)	112.00	30.75	1.08	0
(2, 75, A)	45.33	2.55	1.00	0
(2, 75, B)	146.33	94.03	1.02	0
(2, 75, C)	242.67	636.72	1.07	1
(2, 100, A)	37.00	2.24	1.01	0
(2, 100, B)	188.00	234.44	1.01	0
(2, 100, C)	293.00	1215.17	1.05	2

(n, m, t)	#EQIne	Time (s)	PoS	#TL
(3, 25, A)	17.33	0.79	1.01	0
(3, 25, B)	29.67	1.36	1.02	0
(3, 25, C)	157.33	640.02	1.26	1
(3, 50, A)	67.00	115.06	1.02	0
(3, 50, B)	182.00	627.30	1.01	1
(3, 50, C)	193.67	1800.20	-	3
(3, 75, A)	156.33	1267.78	1.01	2
(3, 75, B)	297.33	1800.01	-	3
(3, 75, C)	179.00	1800.19	-	3
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(2, 100, A)	37.00	2.24	1.01	0
(2, 100, B)	188.00	234.44	1.01	0
(2, 100, C)	293.00	1215.17	1.05	2

(n, m, t)	#EQIne	Time (s)	PoS	#TL
(3, 25, A)	17.33	0.79	1.01	0
(3, 25, B)	29.67	1.36	1.02	0
(3, 25, C)	157.33	640.02	1.26	1
(3, 50, A)	67.00	115.06	1.02	0
(3, 50, B)	182.00	627.30	1.01	1
(3, 50, C)	193.67	1800.20	-	3
(3, 75, A)	156.33	1267.78	1.01	2
(3, 75, B)	297.33	1800.01	-	3
(3, 75, C)	179.00	1800.19	-	3
(3, 100, A)	156.33	1267.78	1.01	2
(3, 100, B)	297.33	1800.01	-	3
(3, 100, C)	179.00	1800.19	-	3

Knapsack Game

(n, m, t)	EQIne	Time (s)	PoS	#TL
(2, 25, A)	10.67	0.08	1.04	0
(2, 25, B)	15.67	0.17	1.02	0
(2, 25, C)	40.00	1.52	1.06	0
(2, 50, A)	15.00	0.22	1.02	0
(2, 50, B)	41.67	1.27	1.01	0
(2, 50, C)	112.00	30.75	1.08	0
(2, 75, A)	45.33	2.55	1.00	0
(2, 75, B)	146.33	94.03	1.02	0
(2, 75, C)	242.67	636.72	1.07	1
(2, 100, A)	37.00	2.24	1.01	0
(2, 100, B)	188.00	234.44	1.01	0
(2, 100, C)	293.00	1215.17	1.05	2

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(3, 100, B)	297.33	1800.01	-	3
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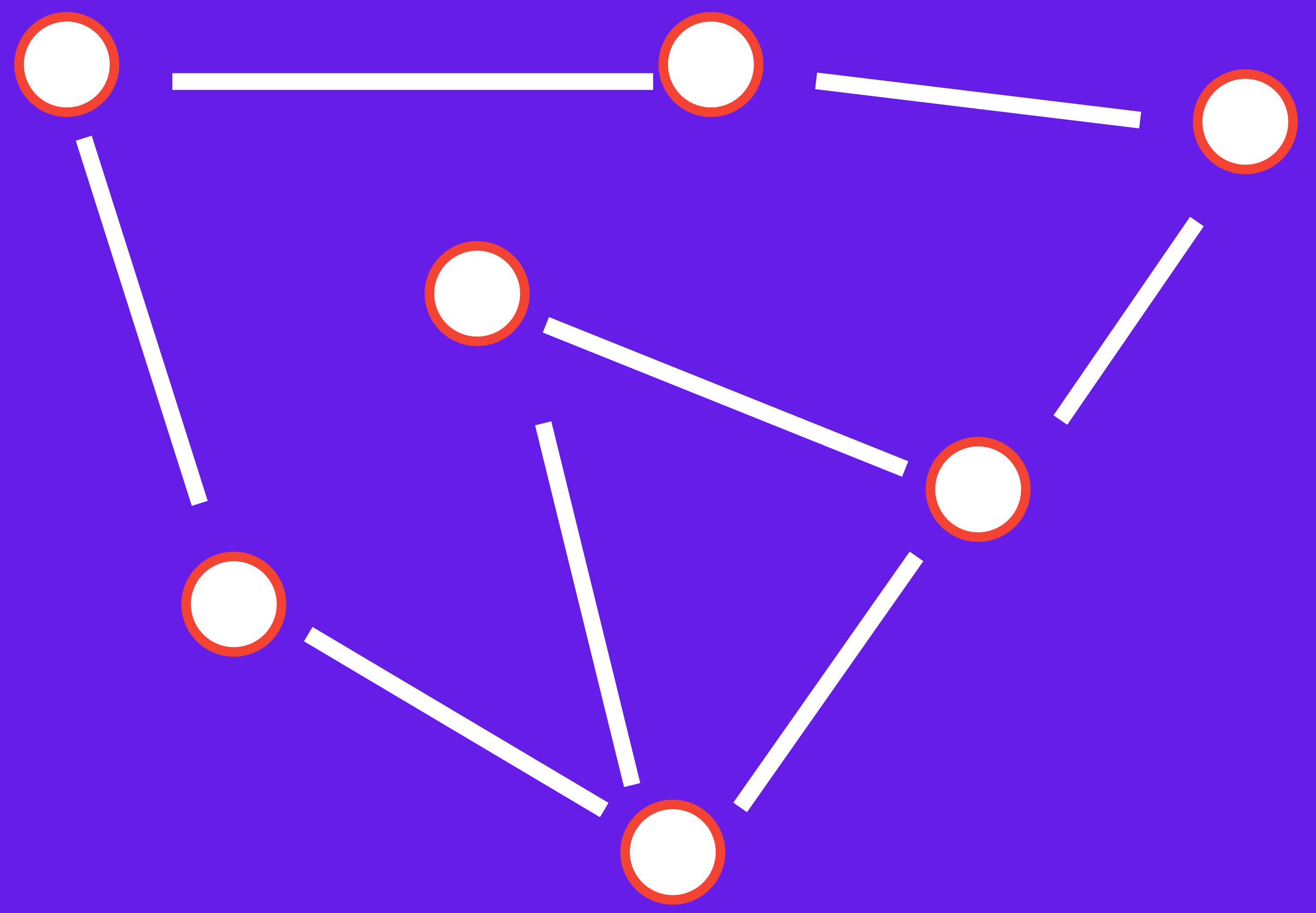
Network Formation Game



(Chen and Roughgarden, 2006; Anshelevich, et al., 2008; Nisan et al., 2008)

Given a graph $G = (V, E)$:

- Each edge $(h, l) \in E : h, l \in V$ has a cost $c_{hl} \in \mathbb{Z}^+$
- Each player i has a weight w^i to go from **a source** s^i **to a sink** t^i
- If more than one player selects an edge, the cost is **split proportionally to each player's weight** (Shapley sharing)

Selecting a PNE with $n \geq 3$ is $\mathcal{NP}$-hard

Network Formation Game

(|V|, |E|)	EQIne	Time (s)	PoS	TL
(50, 99)	5.00	0.07	1.12	0
(100, 206)	9.00	0.13	1.00	0
(150, 308)	9.67	0.47	1.01	0
(200, 416)	18.67	1.85	1.00	0
(250, 517)	68.67	51.55	1.02	0

(|V|, |E|)	EQIne	Time (s)	PoS	TL
(300, 626)	20.33	6.52	1.00	0
(350, 730)	20.67	6.70	1.00	0
(400, 822)	302.00	654.73	1.01	1
(450, 934)	492.00	1200.43	1.01	2
(500, 1060)	40.33	104.80	1.00	0

Network Formation Game

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(50, 99)	5.00	0.07	1.12	0
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Network Formation Game

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(500, 1060)	40.33	104.80	1.00	0

When Nash Meets Stackelberg

Joint work with **Margarida Carvalho, Felipe Feijoo, Andrea Lodi** and **Sriram Sankaranarayanan**

3

🧙🏾‍♂️ and 🧚🏼‍♂️

want to change life

They want to sell bagels for a living

WizardMount Bagels©

Simultaneous
Game

St Fairy Bagels©

Magicville taxes their bagels

Since ovens are *polluting* the city’s air

WizardMount Bagels©

Simultaneous
Game

St Fairy Bagels©

Sequential
“Stackelberg” Game

Simultaneous
Game

WizardMount Bagels©

St Fairy Bagels©

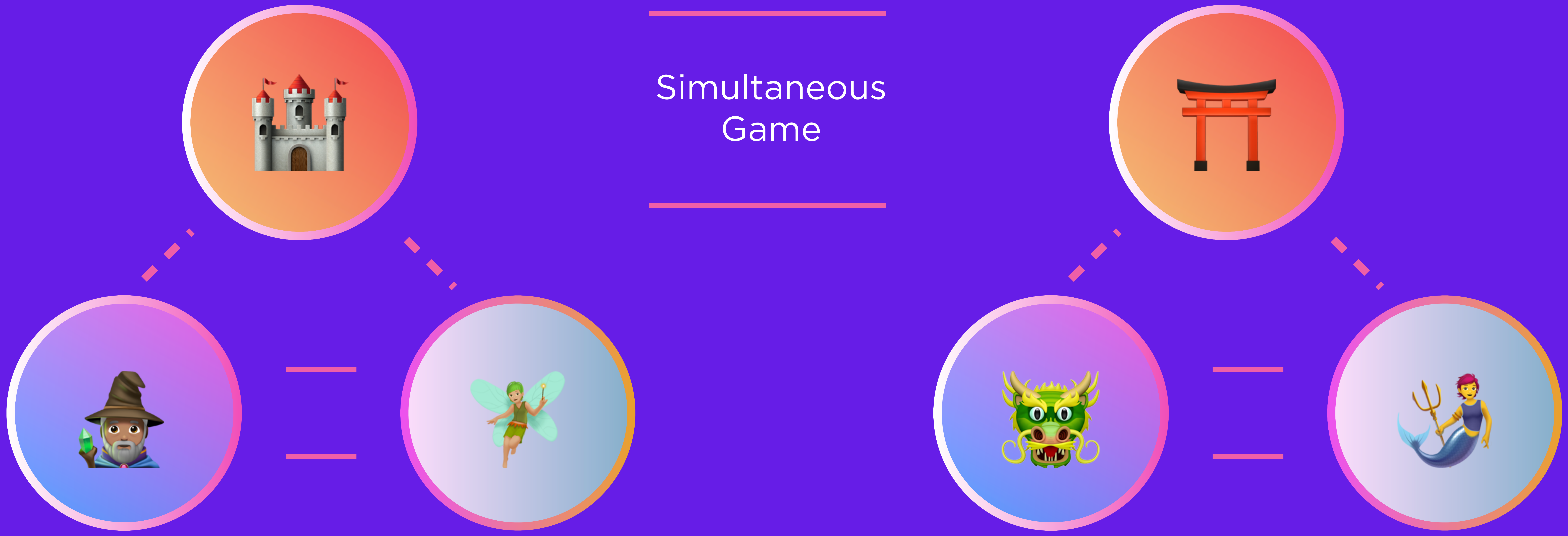
Magicville

Simultaneous Game

Witchtown

Magicville

Witchtown



Cities can import, export (or block imports and exports) of bagels
Tax their producers

We define them as Nash Games among Stackelberg Players (*NASPs*)

We define them as Nash Games among Stackelberg Players (*NASPs*)

We define them as Nash Games among Stackelberg Players (*NASPs*)

Bagels are units of energy

We define them as Nash Games among Stackelberg Players (*NASPs*)

Bagels are units of energy

Cities are regulatory agencies

Contributions

Complexity

It is Σ_2^p-hard to determine a MNE/PNE, in general

Algorithms

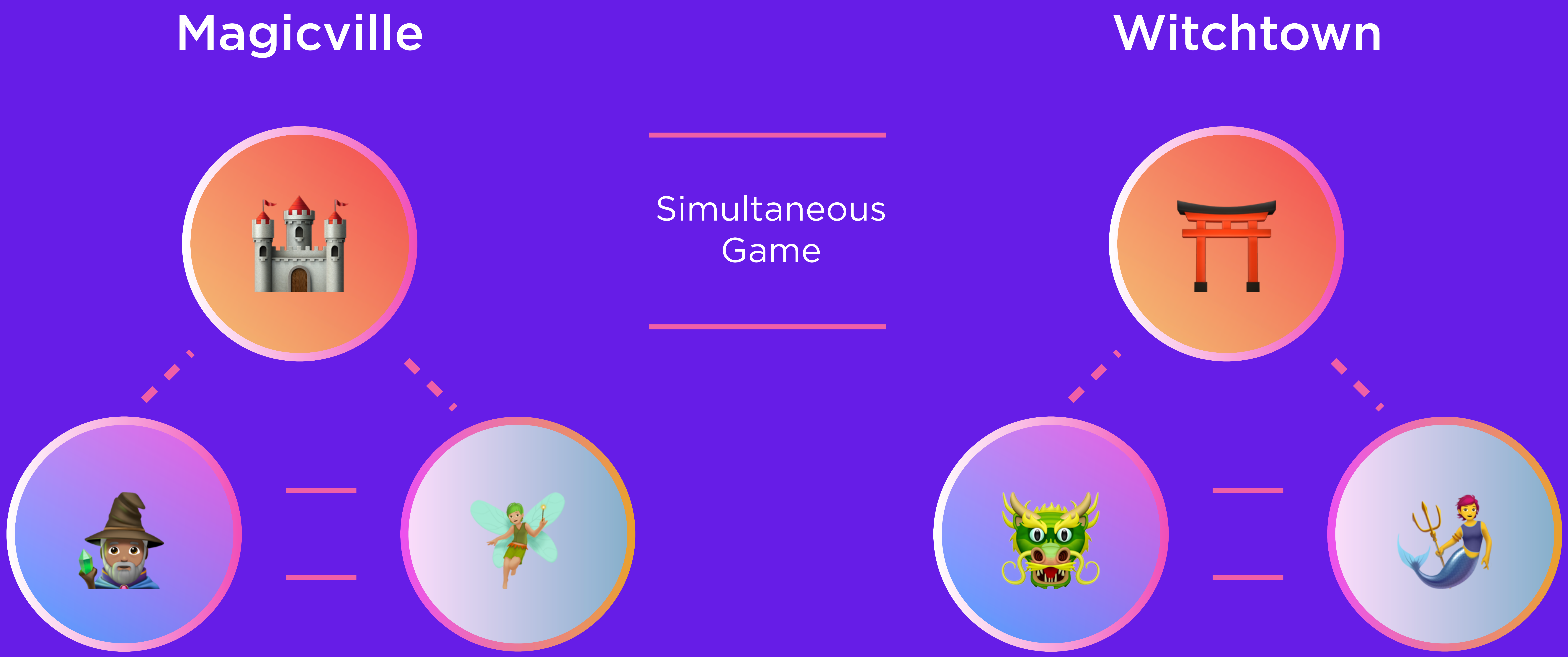
A **full enumeration scheme**, and an inner **approximation scheme**

Insights

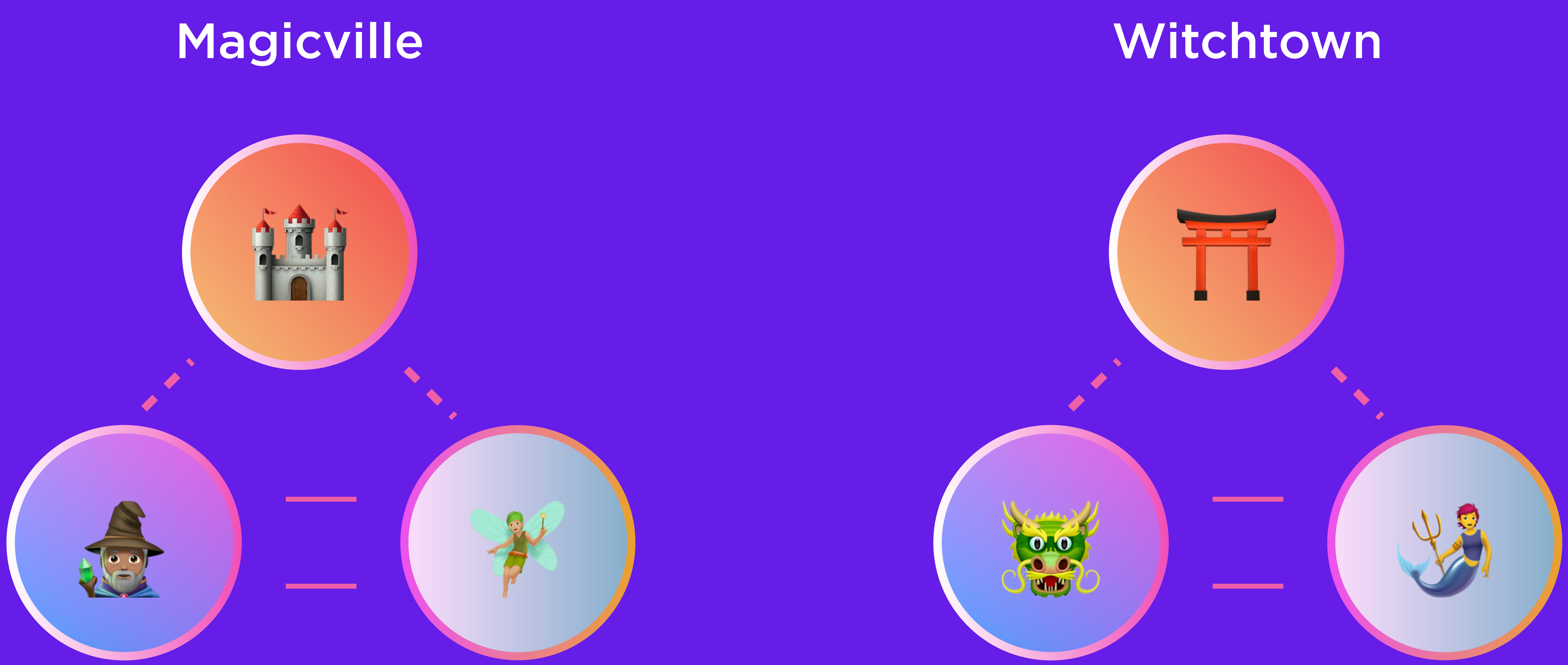
Energy market tests, with **Chilean-Argentinean case study**

Magicville

Reformulate each Stackelberg game as a single-level Optimization problem



Then, the game is an **RBG,** if objectives are compatible



Among the reformulated bilevel programs, namely the ***real players***

Magicville

$$\max_{x^i} \{ (c^i)^\top x^i + (x^{-i})^\top C^i x^i : \underline{x^i \in \mathscr{F}^i} \}$$

The reformulated feasible region includes the KKT for the followers' problems

$$\mathcal{F}^i = \left\{ \begin{array}{l} A^i x^i \leq b^i \\ z^i = M^i x^i + q^i \\ x^i \geq 0, z^i \geq 0 \end{array} \right\} \bigcap_{j \in \mathcal{C}^i} \left(\{ z^i_j = 0 \} \cup \{ x^i_j = 0 \} \right).$$

Magicville

$$\max_{x^i} \{ (c^i)^\top x^i + (x^{-i})^\top C^i x^i : x^i \in \mathcal{F}^i \}$$

Algorithms

Fully enumerate cl conv($\mathcal{F}^i$)

Inner approximate cl conv($\mathcal{F}^i$) (dual to CnP)

Magicville

Algorithms

	Time (s)	# TL
Fully enumerate cl conv$(\mathscr{F}^i)$	120.2	9/149
Inner approximate cl conv$(\mathscr{F}^i)$	3.73	0/149

ZERO: Playing MPGs

Joint work with **Margarida Carvalho, Andrea Lodi** and **Sriram Sankaranarayanan**

4

🧙🏾‍♂️ and 🧚🏼‍♂️

should really use ZERO to plan their lives.

ZERO™

Everything we dealt with (and more) is currently implemented in a software called ZERO

- **Modular and extensible** C++ library
- Models, abstracts, and solves *RBGs, IPGs, NASPs, simple Bilevel problems…*
- Builds like a library that can be **integrated** in third-party projects
- Supports explicit modeling for energy trade markets

Plan for future developments:

- A plan to scale up the project and **industrial partnerships (ongoing)**
- Integration with SCIP Optimization Suite

An Open Source Solver

```
Models::IPG::IPG IPG_Model(&GurobiEnv, IPG_Instance);
// Select the equilibrium to compute a Nash Equilibrium
IPG_Model.setAlgorithm(Data::IPG::Algorithms::CutAndPlay);
// Extra parameters
IPG_Model.setDeviationTolerance(3e-4);
IPG_Model.setNumThreads(8);
IPG_Model.setLCPAlgorithm(Data::LCP::Algorithms::PATH);

// Lock the model
IPG_Model.finalize();
// Run!
IPG_Model.findNashEq();
```

Conclusions


MPGs
IPGs
RBGs
Finite Games

We proposed

The Cut-And-Play

Algorithms

When Nash Meets Stackelberg

Algorithms

Modeling

Insights

ZERO Regrets

Algorithms

Efficiency

ZERO: Playing MPGs

Algorithms

Ideas

Methodology

Development of more **theoretical and algorithmic machinery** to compute Nash equilibria

Exploit ***different solution concepts***, and focus on the ***selection*** of such solutions.

Understand ***rational behavior through inequalities***

Practical

Better understanding of ***MPGs and their application domains.*** For instance, ***energy markets***

Ethical

Companies, governments, and in general, organizations are likely to solve optimization problems. Trade-off ***selfishness and social good***


and they lived happily ever after.

Mathematical Programming Games

CANADA
EXCELLENCE
RESEARCH
CHAIR

DATA SCIENCE
FOR REAL-TIME
DECISION-MAKING

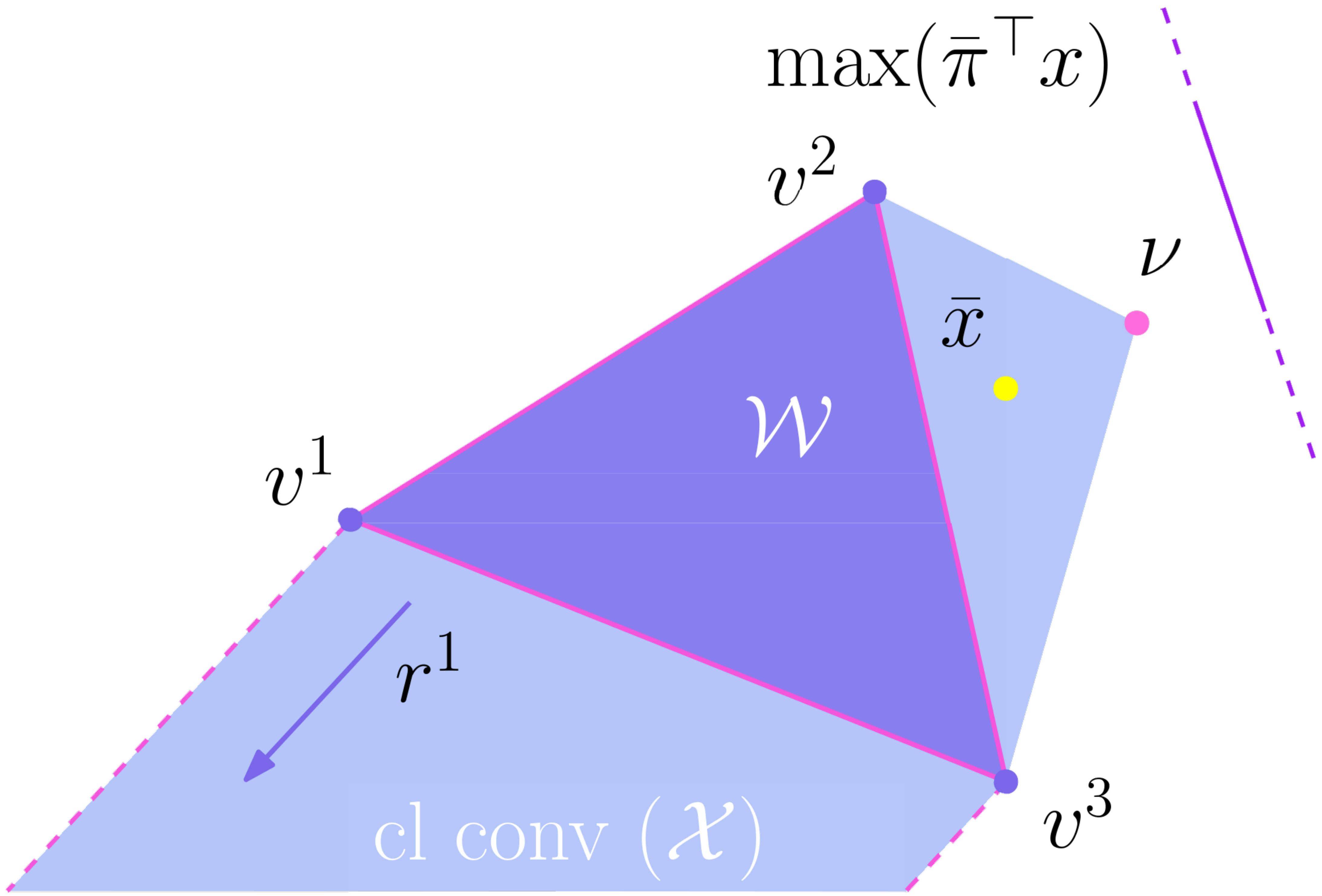
POLYTECHNIQUE
MONTRÉAL

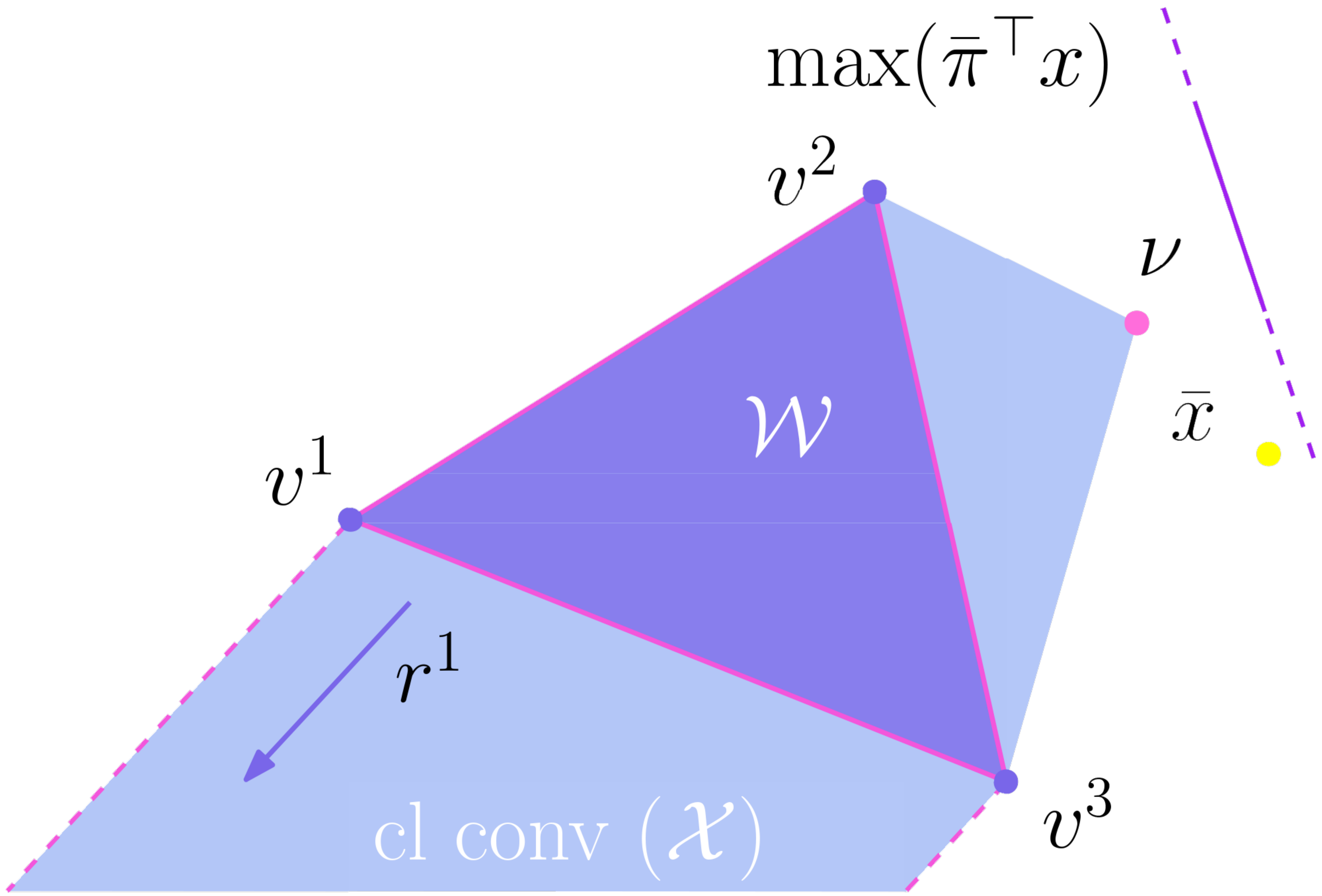
TECHNOLOGICAL
UNIVERSITY

Appendix

Cut-And-Play

Enhanced Separation Oracle



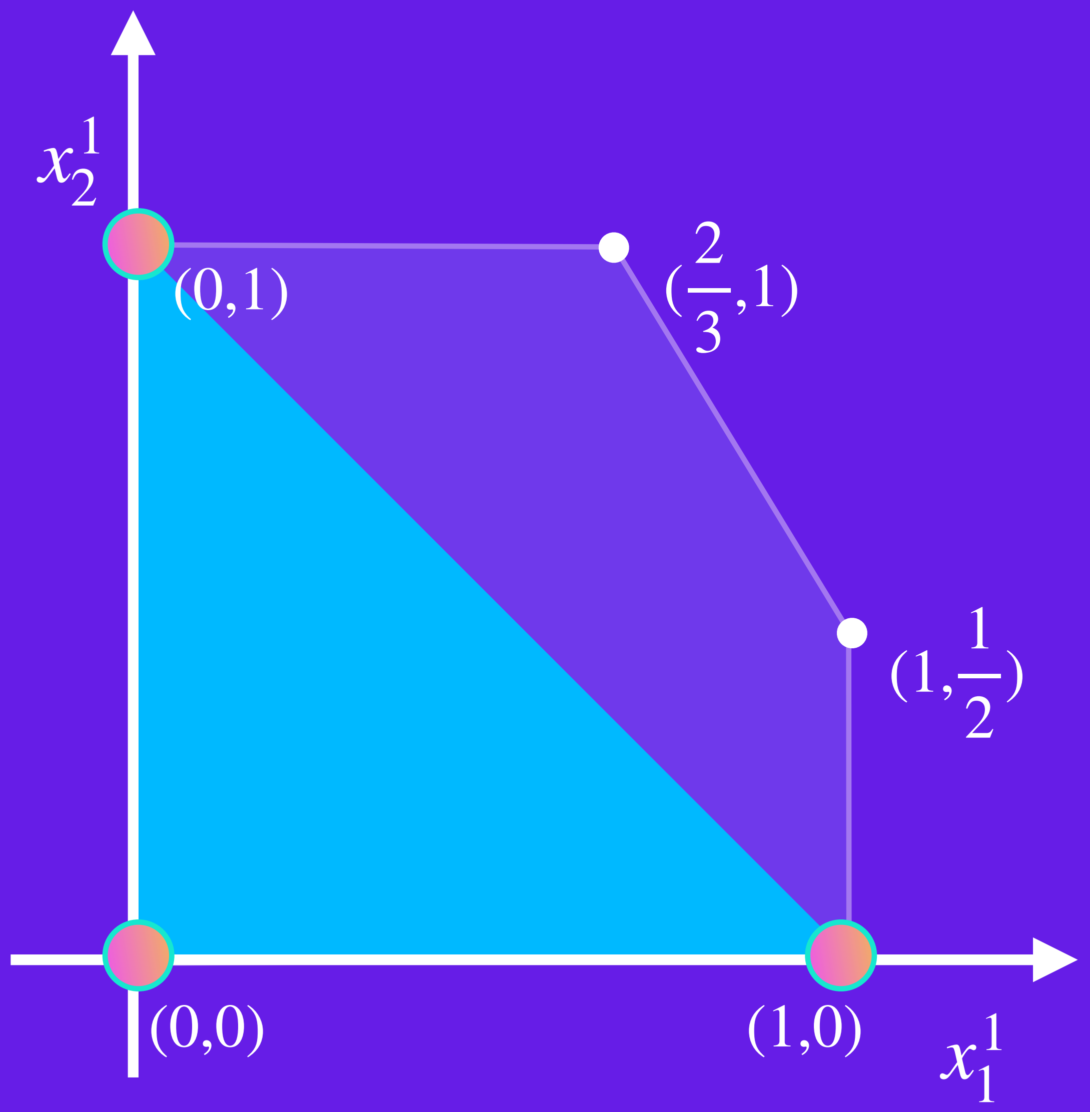


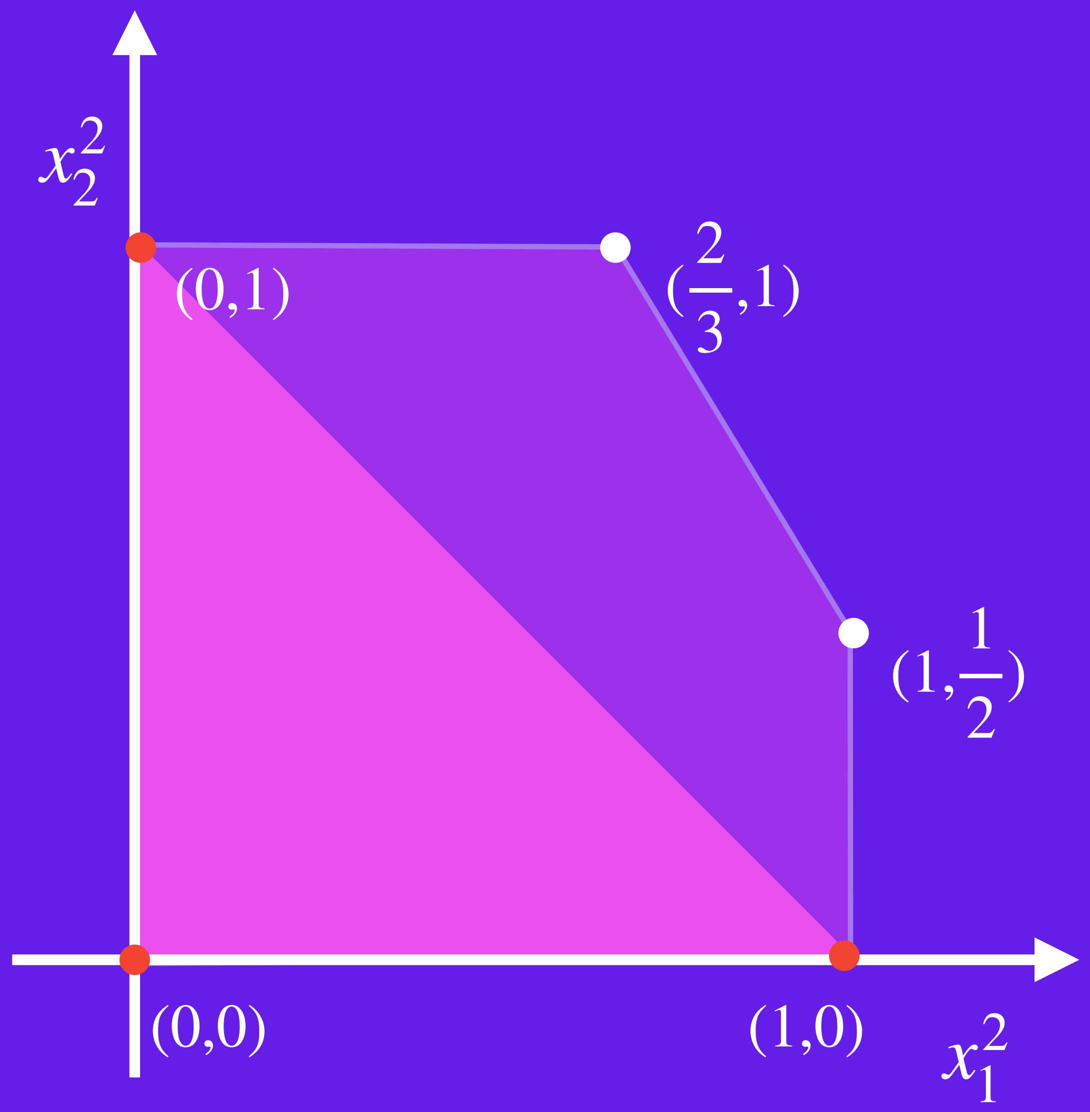
ZERO Regrets

$$\max_{x^1} \quad 6x_1^1 + x_2^1 - 4x_1^1x_1^2 + 3x_2^1x_2^2$$
$$\text{s.t.} \quad 3x_1^1 + 2x_2^1 \leq 4$$
$$x^1 \in \{0,1\}^2$$

$$\max_{x^2} \quad 4x_1^2 + 2x_2^2 - x_1^2x_1^1 - x_2^2x_2^1$$
$$\text{s.t.} \quad 3x_1^2 + 2x_2^2 \leq 4$$
$$x^2 \in \{0,1\}^2$$

Best-responses

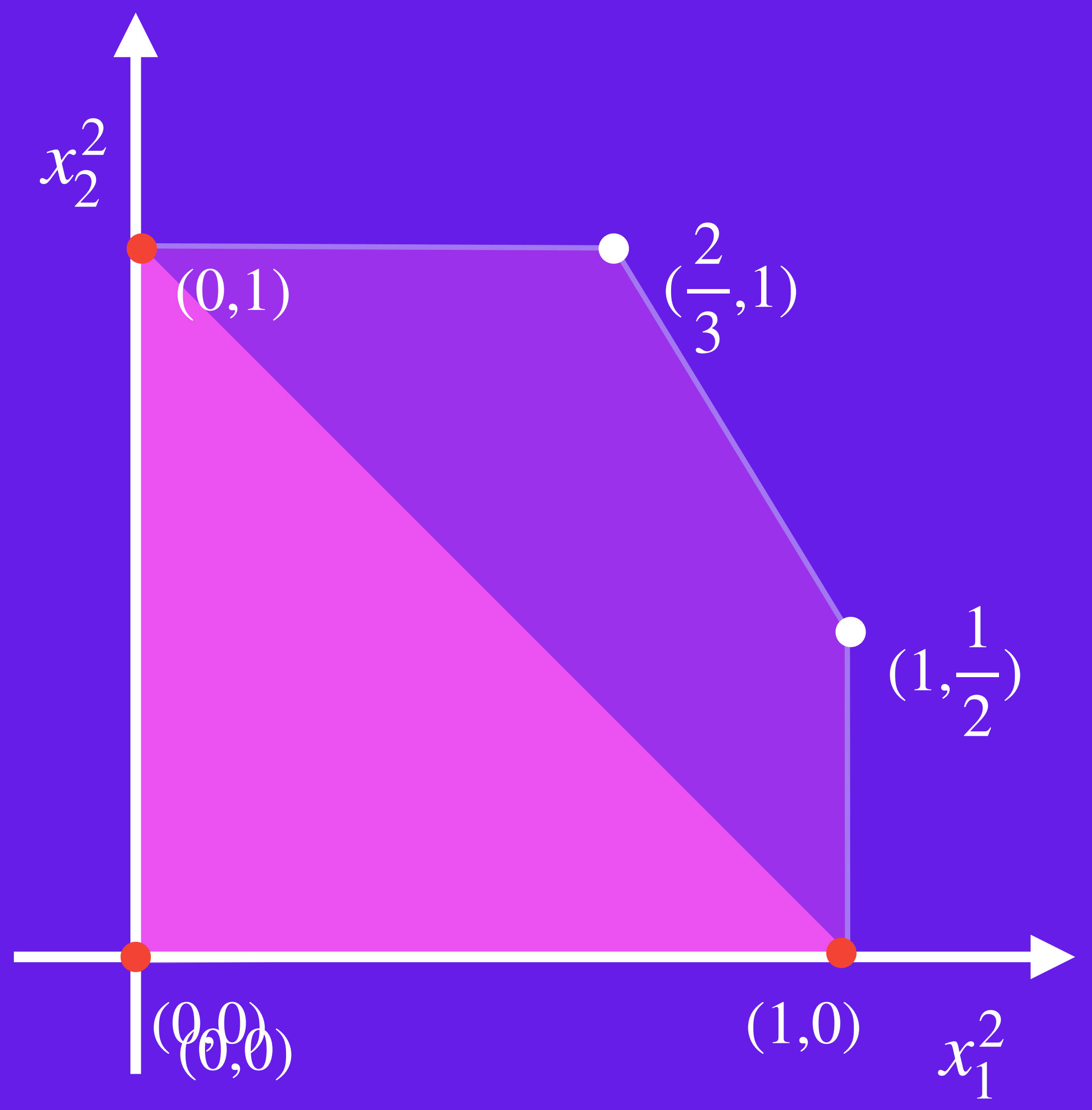




$$\max_{x^2} \quad 4x_1^2 + 2x_2^2 \; - x_1^2 x_1^1 - x_2^2 x_2^1$$

$$\text{s.t.} \quad 3x_1^2 + 2x_2^2 \leq 4$$

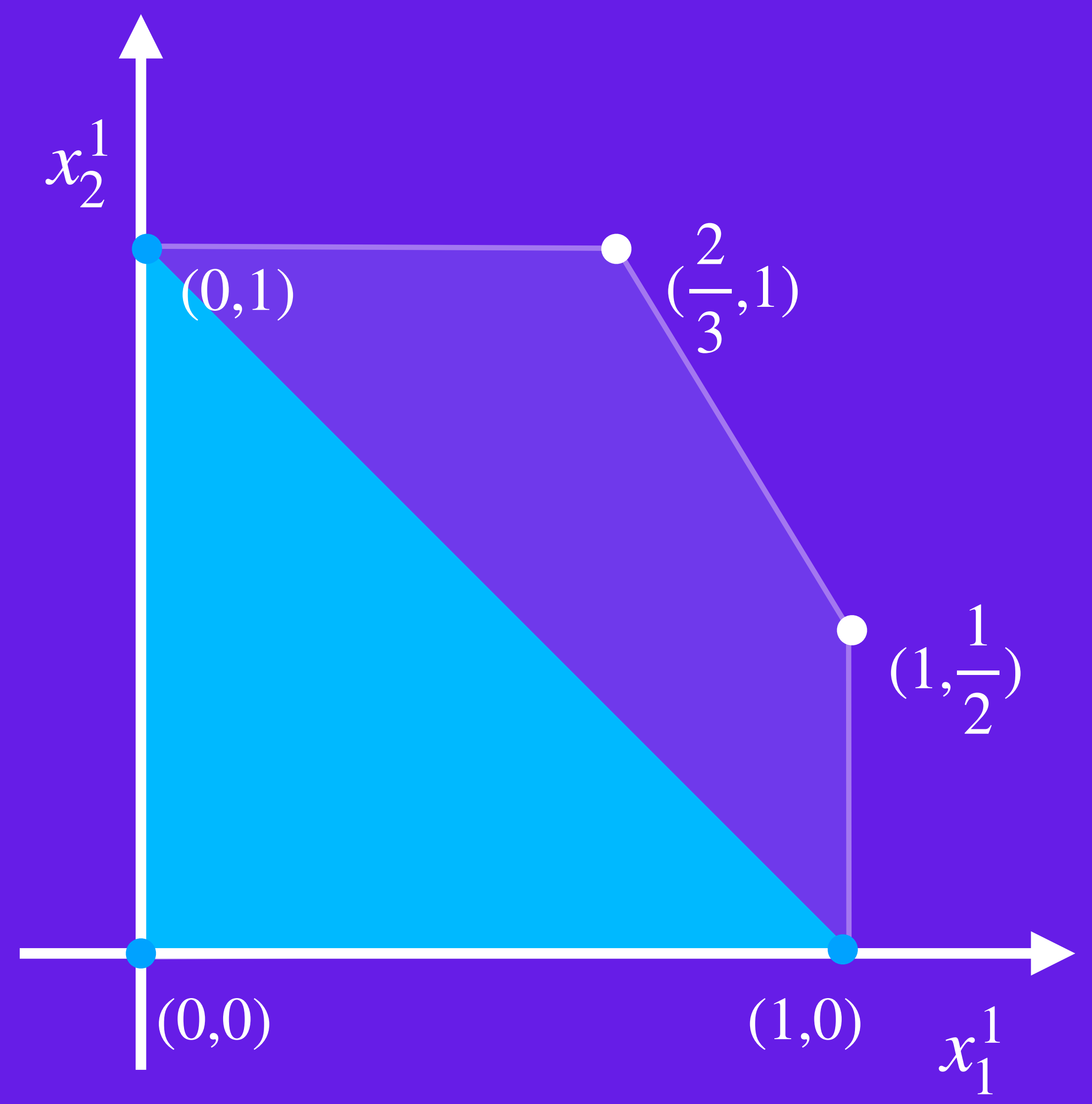
$$x^2 \in \{0,1\}^2$$



(0,0)

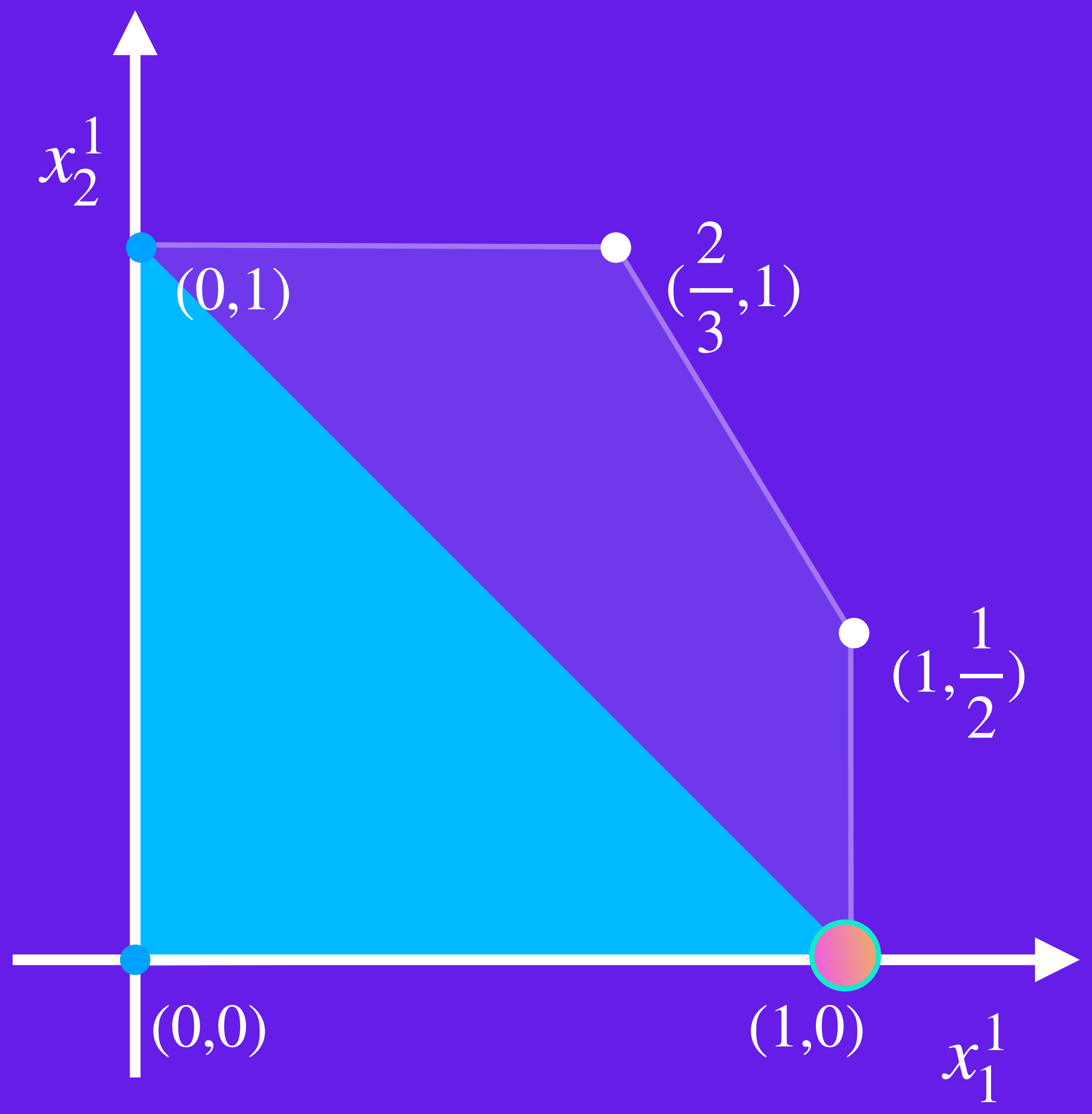
		x^2		
		(0,0)	(1,0)	(0,1)
x^1	(0,0)	0 0	0 4	0 2
	(1,0)	6 0	2 3	6 2
	(0,1)	1 0	1 2	-3 1

$$\max_{x^1} \quad 6x_1^1 + x_2^1 - 4x_1^1x_1^2 + 3x_2^1x_2^2$$
$$\text{s.t.} \quad 3x_1^1 + 2x_2^1 \leq 4$$
$$x^1 \in \{0,1\}^2$$

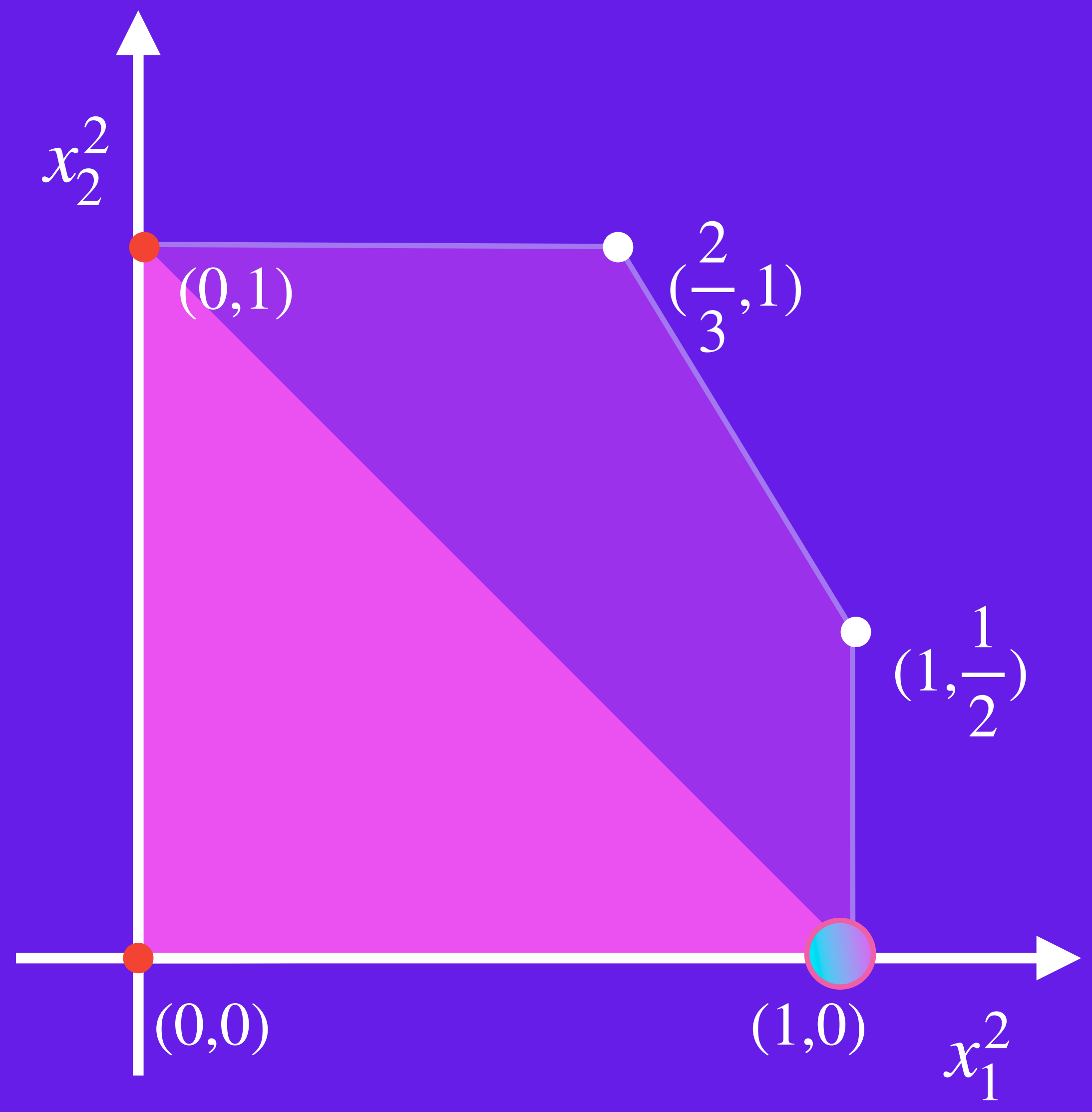


x^1 \ x^2	(0,0)	(1,0)	(0,1)
(0,0)	0 0	0 4	0 2
(1,0)	6 0	2 3	6 2
(0,1)	1 0	1 2	-3 1

$$\max_{x^1} \quad 6x_1^1 + x_2^1 - 4x_1^1 x_1^2 + 3x_2^1 x_2^2$$
$$\text{s.t.} \quad 3x_1^1 + 2x_2^1 \leq 4$$
$$x^1 \in \{0,1\}^2$$



$$\max_{x^2} \quad 4x_1^2 + 2x_2^2 - x_1^2 x_1^1 - x_2^2 x_2^1$$
$$\text{s.t.} \quad 3x_1^2 + 2x_2^2 \leq 4$$
$$x^2 \in \{0,1\}^2$$



Their products **interact**!

$$\max_{x^1} \quad x_1^1 + 8x_2^1 + 4x_1^1 x_1^2 - 7x_2^1 x_2^2$$

$$\text{s.t.} \quad 7x_1^1 + 2x_2^1 \leq 7$$

$$x^1 \in \{0,1\}^2$$

$$\max_{x^2} \quad 5x_1^2 + 10x_2^2 + 5x_1^2 x_1^1 - 5x_2^2 x_2^1$$

$$\text{s.t.} \quad 9x_1^2 + 7x_2^2 \leq 12$$

$$x^2 \in \{0,1\}^2$$

$$\max_{x^1} \quad x_1^1 + 8x_2^1 + 4x_1^1x_1^2 - 7x_2^1x_2^2$$

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$$\max_{x^2} \quad 5x_1^2 + 10x_2^2 + 5x_1^2x_1^1 - 5x_2^2x_2^1$$

$$\text{s.t.} \quad 9x_1^2 + 7x_2^2 \leq 12$$

$$x^2 \in \{0,1\}^2$$

$$\mathcal{Q} = \max_{x,z} \quad 1x_1^1 + 8x_2^1 + 5x_1^2 + 10x_2^2 - 9z_1 - 12z_2$$

$$\text{s.t.} \quad 7x_1^1 + 2x_2^1 \leq 7, \quad 9x_1^2 + 7x_2^2 \leq 12$$

$$x_j^1 + x_j^2 - 1 \leq z_j, \quad x_j^1 \geq z_j, \quad x_j^2 \geq z_j \quad j = 1,2$$

$$x_j^1, x_j^2, z_j \in \{0,1\} \quad i = 1,2$$

$$\max_{x^1} \quad x_1^1 + 8x_2^1 + 4x_1^1 x_1^2 - 7x_2^1 x_2^2$$

$$7x_1^1 + 2x_2^1 \leq 7$$

$$x^1 \in \{0,1\}^2$$

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$$x_j^1, x_j^2, z_j \in \{0,1\} \quad j = 1,2$$

x_1^1	x_2^1	x_1^2	x_2^2	z_1	z_2
1.0	0.0	1.0	0.0	1.0	0.0
0.0	1.0	0.0	1.0	0.0	1.0
0.0	0.0	0.0	0.0	0.0	0.0
1.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	1.0	0.0	0.0	0.0
1.0	0.0	0.0	1.0	0.0	0.0
0.0	0.0	0.0	1.0	0.0	0.0
0.0	1.0	0.0	0.0	0.0	0.0
0.0	1.0	1.0	0.0	0.0	0.0

$$\max_{x^1} \quad x_1^1 + 8x_2^1 + 4x_1^1x_1^2 - 7x_2^1x_2^2$$

$$7x_1^1 + 2x_2^1 \leq 7$$

$$x^1 \in \{0,1\}^2$$

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$$\text{s.t.} \quad 9x_1^2 + 7x_2^2 \leq 12$$

$$x^2 \in \{0,1\}^2$$

$$\mathcal{Q} = \max_{x,z} \quad 1x_1^1 + 8x_2^1 + 5x_1^2 + 10x_2^2 - 9z_1 - 12z_2$$

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$$x_j^1 + x_j^2 - 1 \leq z_j, \quad x_j^1 \geq z_j, \quad x_j^2 \geq z_j \quad j = 1,2$$

$$x_j^1, x_j^2, z_j \in \{0,1\} \quad j = 1,2$$

x_1^1	x_2^1	x_1^2	x_2^2	z_1	z_2
1.0	0.0	1.0	0.0	1.0	0.0
0.0	1.0	0.0	1.0	0.0	1.0
0.0	0.0	0.0	0.0	0.0	0.0
1.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	1.0	0.0	0.0	0.0
1.0	0.0	0.0	1.0	0.0	0.0
0.0	0.0	0.0	1.0	0.0	0.0
0.0	1.0	0.0	0.0	0.0	0.0
0.0	1.0	1.0	0.0	0.0	0.0

$$\max_{x^1} \quad x_1^1 + 8x_2^1 + 4x_1^1x_1^2 - 7x_2^1x_2^2$$
$$7x_1^1 + 2x_2^1 \leq 7$$
$$x^1 \in \{0,1\}^2$$

$$\max_{x^2} \quad 5x_1^2 + 10x_2^2 + 5x_1^2x_1^1 - 5x_2^2x_2^1$$
$$\text{s.t.} \quad 9x_1^2 + 7x_2^2 \leq 12$$
$$x^2 \in \{0,1\}^2$$

$$\mathcal{Q} = \max_{x,z} \quad 1x_1^1 + 8x_2^1 + 5x_1^2 + 10x_2^2 - 9z_1 - 12z_2$$
$$\text{s.t.} \quad 7x_1^1 + 2x_2^1 \leq 7, \quad 9x_1^2 + 7x_2^2 \leq 12$$
$$x_j^1 + x_j^2 - 1 \leq z_j, \quad x_j^1 \geq z_j, \quad x_j^2 \geq z_j \quad j = 1,2$$
$$x_j^1, x_j^2, z_j \in \{0,1\} \quad j = 1,2$$

x_1^1	x_2^1	x_1^2	x_2^2	z_1	z_2
1.0	0.0	1.0	0.0	1.0	0.0
0.0	1.0	0.0	1.0	0.0	1.0
0.0	0.0	0.0	0.0	0.0	0.0
1.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	1.0	0.0	0.0	0.0
1.0	0.0	0.0	1.0	0.0	0.0
0.0	0.0	0.0	1.0	0.0	0.0
0.0	1.0	0.0	0.0	0.0	0.0
0.0	1.0	1.0	0.0	0.0	0.0

$$(x_1^1, x_2^1) = (0,1) \quad 1x_1^1 + 8x_2^1 + 7x_2^2 + 4z_1 - 7z_2 \geq 8$$
$$(x_1^1, x_2^1) = (1,0) \quad 1x_1^1 + 8x_2^1 - 4x_1^2 + 4z_1 - 7z_2 \geq 1$$
$$(x_1^2, x_2^2) = (1,0) \quad 5x_1^1 + 10x_2^1 - 4x_1^2 + 5z_1 - 5z_2 \geq 5$$
$$(x_1^2, x_2^2) = (0,1) \quad 5x_1^1 + 10x_2^1 + 7x_2^2 + 5z_1 - 5z_2 \geq 10$$

$$\max_{x^1} \quad x_1^1 + 8x_2^1 + 4x_1^1x_1^2 - 7x_2^1x_2^2$$
$$7x_1^1 + 2x_2^1 \le 7$$
$$x^1 \in \{0,1\}^2$$

$$\max_{x^2} \quad 5x_1^2 + 10x_2^2 + 5x_1^2x_1^1 - 5x_2^2x_2^1$$
$$\text{s.t.} \quad 9x_1^2 + 7x_2^2 \le 12$$
$$x^2 \in \{0,1\}^2$$

$$\mathcal{Q} = \max_{x,z} \quad 1x_1^1 + 8x_2^1 + 5x_1^2 + 10x_2^2 - 9z_1 - 12z_2$$
$$\text{s.t.} \quad 7x_1^1 + 2x_2^1 \le 7, \quad 9x_1^2 + 7x_2^2 \le 12$$
$$x_j^1 + x_j^2 - 1 \le z_j, \quad x_j^1 \ge z_j, \quad x_j^2 \ge z_j \quad j = 1,2$$
$$x_j^1, x_j^2, z_j \in \{0,1\} \quad j = 1,2$$

x_1^1	x_2^1	x_1^2	x_2^2	z_1	z_2
~~1.0~~	~~0.0~~	~~1.0~~	~~0.0~~	~~1.0~~	~~0.0~~
0.0	1.0	0.0	1.0	0.0	1.0
~~0.0~~	~~0.0~~	~~0.0~~	~~0.0~~	~~0.0~~	~~0.0~~
~~1.0~~	~~0.0~~	~~0.0~~	~~0.0~~	~~0.0~~	~~0.0~~
~~0.0~~	~~0.0~~	~~1.0~~	~~0.0~~	~~0.0~~	~~0.0~~
1.0	0.0	0.0	1.0	0.0	0.0
~~0.0~~	~~0.0~~	~~0.0~~	~~1.0~~	~~0.0~~	~~0.0~~
~~0.0~~	~~1.0~~	~~0.0~~	~~0.0~~	~~0.0~~	~~0.0~~
0.0	1.0	1.0	0.0	0.0	0.0

$(x_1^1, x_2^1) = (0,1) \quad 1x_1^1 + 8x_2^1 + 7x_2^2 + 4z_1 - 7z_2 \ge 8$

$(x_1^1, x_2^1) = (1,0) \quad 1x_1^1 + 8x_2^1 - 4x_1^2 + 4z_1 - 7z_2 \ge 1$

$(x_1^2, x_2^2) = (1,0) \quad 5x_1^1 + 10x_2^1 - 4x_1^2 + 5z_1 - 5z_2 \ge 5$

$(x_1^2, x_2^2) = (0,1) \quad 5x_1^1 + 10x_2^1 + 7x_2^2 + 5z_1 - 5z_2 \ge 10$

$$\max_{x^1} \quad x^1_1 + 8x^1_2 + 4x^1_1x^2_1 - 7x^1_2x^2_2$$

$$7x^1_1 + 2x^1_2 \le 7$$

$$x^1 \in \{0,1\}^2$$

$$\max_{x^2} \quad 5x^2_1 + 10x^2_2 + 5x^2_1x^1_1 - 5x^2_2x^1_2$$

$$\text{s.t.} \quad 9x^2_1 + 7x^2_2 \le 12$$

$$x^2 \in \{0,1\}^2$$

$$\mathcal{Q} = \max_{x,z} \quad 1x^1_1 + 8x^1_2 + 5x^2_1 + 10x^2_2 - 9z_1 - 12z_2$$

$$\text{s.t.} \quad 7x^1_1 + 2x^1_2 \le 7, \quad 9x^2_1 + 7x^2_2 \le 12$$

$$x^1_j + x^2_j - 1 \le z_j, \quad x^1_j \ge z_j, \quad x^2_j \ge z_j \quad j = 1,2$$

$$x^1_j, x^2_j, z_j \in \{0,1\} \quad j = 1,2$$

x^1_1	x^1_2	x^2_1	x^2_2	z_1	z_2
~~1.0~~	~~0.0~~	~~1.0~~	~~0.0~~	~~1.0~~	~~0.0~~
0.0	1.0	0.0	1.0	0.0	1.0
~~0.0~~	~~0.0~~	~~0.0~~	~~0.0~~	~~0.0~~	~~0.0~~
~~1.0~~	~~0.0~~	~~0.0~~	~~0.0~~	~~0.0~~	~~0.0~~
~~0.0~~	~~0.0~~	~~1.0~~	~~0.0~~	~~0.0~~	~~0.0~~
1.0	0.0	0.0	1.0	0.0	0.0
~~0.0~~	~~0.0~~	~~0.0~~	~~1.0~~	~~0.0~~	~~0.0~~
~~0.0~~	~~1.0~~	~~0.0~~	~~0.0~~	~~0.0~~	~~0.0~~
0.0	1.0	1.0	0.0	0.0	0.0

$$(x^1_1, x^1_2) = (0,1) \quad 1x^1_1 + 8x^1_2 + 7x^2_2 + 4z_1 - 7z_2 \ge 8$$

$$(x^1_1, x^1_2) = (1,0) \quad 1x^1_1 + 8x^1_2 - 4x^2_1 + 4z_1 - 7z_2 \ge 1$$

$$(x^2_1, x^2_2) = (1,0) \quad 5x^1_1 + 10x^1_2 - 4x^2_1 + 5z_1 - 5z_2 \ge 5$$

$$(x^2_1, x^2_2) = (0,1) \quad 5x^1_1 + 10x^1_2 + 7x^2_2 + 5z_1 - 5z_2 \ge 10$$

 

$$\max_{x^1} \quad x_1^1 + 8x_2^1 + 4x_1^1x_1^2 - 7x_2^1x_2^2$$
$$7x_1^1 + 2x_2^1 \le 7$$
$$x^1 \in \{0,1\}^2$$

$$\max_{x^2} \quad 5x_1^2 + 10x_2^2 + 5x_1^2x_1^1 - 5x_2^2x_2^1$$
$$\text{s.t.} \quad 9x_1^2 + 7x_2^2 \le 12$$
$$x^2 \in \{0,1\}^2$$

$$\mathcal{Q} = \max_{x,z} \quad 1x_1^1 + 8x_2^1 + 5x_1^2 + 10x_2^2 - 9z_1 - 12z_2$$
$$\text{s.t.} \quad 7x_1^1 + 2x_2^1 \le 7, \quad 9x_1^2 + 7x_2^2 \le 12$$
$$x_j^1 + x_j^2 - 1 \le z_j, \quad x_j^1 \ge z_j, \quad x_j^2 \ge z_j \quad j = 1,2$$
$$x_j^1, x_j^2, z_j \in \{0,1\} \quad j = 1,2$$

x_1^1	x_2^1	x_1^2	x_2^2	z_1	z_2
1.0	0.0	1.0	0.0	1.0	0.0
0.0	1.0	0.0	1.0	0.0	1.0
0.0	0.0	0.0	0.0	0.0	0.0
1.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	1.0	0.0	0.0	0.0
1.0	0.0	0.0	1.0	0.0	0.0
0.0	0.0	0.0	1.0	0.0	0.0
0.0	1.0	0.0	0.0	0.0	0.0
0.0	1.0	1.0	0.0	0.0	0.0

$$(x_1^1, x_2^1) = (0,1) \quad 1x_1^1 + 8x_2^1 + 7x_2^2 + 4z_1 - 7z_2 \ge 8$$
$$(x_1^1, x_2^1) = (1,0) \quad 1x_1^1 + 8x_2^1 - 4x_1^2 + 4z_1 - 7z_2 \ge 1$$
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$$\max_{x^1} \quad x_1^1 + 8x_2^1 + 4x_1^1x_1^2 - 7x_2^1x_2^2$$

$$7x_1^1 + 2x_2^1 \leq 7$$

$$x^1 \in \{0,1\}^2$$

$$\max_{x^2} \quad 5x_1^2 + 10x_2^2 + 5x_1^2x_1^1 - 5x_2^2x_2^1$$

$$\text{s.t.} \quad 9x_1^2 + 7x_2^2 \leq 12$$

$$x^2 \in \{0,1\}^2$$

x_1^1	x_2^1	x_1^2	x_2^2	z_1	z_2
1.0	0.0	1.0	0.0	1.0	0.0
0.0	1.0	0.0	1.0	0.0	1.0
0.0	0.0	0.0	0.0	0.0	0.0
1.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	1.0	0.0	0.0	0.0
1.0	0.0	0.0	1.0	0.0	0.0
0.0	0.0	0.0	1.0	0.0	0.0
0.0	1.0	0.0	0.0	0.0	0.0
0.0	1.0	1.0	0.0	0.0	0.0

$(x_1^1, x_2^1) = (0,1) \qquad 1x_1^1 + 8x_2^1 + 7x_2^2 + 4z_1 - 7z_2 \geq 8$

$(x_1^1, x_2^1) = (1,0) \qquad 1x_1^1 + 8x_2^1 - 4x_1^2 + 4z_1 - 7z_2 \geq 1$

$(x_1^2, x_2^2) = (1,0) \qquad 5x_1^1 + 10x_2^1 - 4x_1^2 + 5z_1 - 5z_2 \geq 5$

$(x_1^2, x_2^2) = (0,1) \qquad 5x_1^1 + 10x_2^1 + 7x_2^2 + 5z_1 - 5z_2 \geq 10$

$$\max_{x^1} \quad x_1^1 + 8x_2^1 + 4x_1^1 x_1^2 - 7x_2^1 x_2^2$$

$$7x_1^1 + 2x_2^1 \leq 7$$

$$x^1 \in \{0,1\}^2$$

$$\max_{x^2} \quad 5x_1^2 + 10x_2^2 + 5x_1^2 x_1^1 - 5x_2^2 x_2^1$$

$$\text{s.t.} \quad 9x_1^2 + 7x_2^2 \leq 12$$

$$x^2 \in \{0,1\}^2$$

$$\mathcal{E}$$

$$x_1^1 \geq 0 \qquad x_1^2 \geq 0$$

$$x_1^1 + x_2^1 = 1 \qquad x_1^2 + x_2^2 = 1$$

$$x_1^1 + x_1^2 + z_2 = 1 \qquad x_1^1 + x_1^2 \leq 1$$

$$z_1 = 0$$

x_1^1	x_2^1	x_1^2	x_2^2	z_1	z_2
1.0	0.0	1.0	0.0	1.0	0.0
0.0	1.0	0.0	1.0	0.0	1.0
0.0	0.0	0.0	0.0	0.0	0.0
1.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	1.0	0.0	0.0	0.0
1.0	0.0	0.0	1.0	0.0	0.0
0.0	0.0	0.0	1.0	0.0	0.0
0.0	1.0	0.0	0.0	0.0	0.0
0.0	1.0	1.0	0.0	0.0	0.0

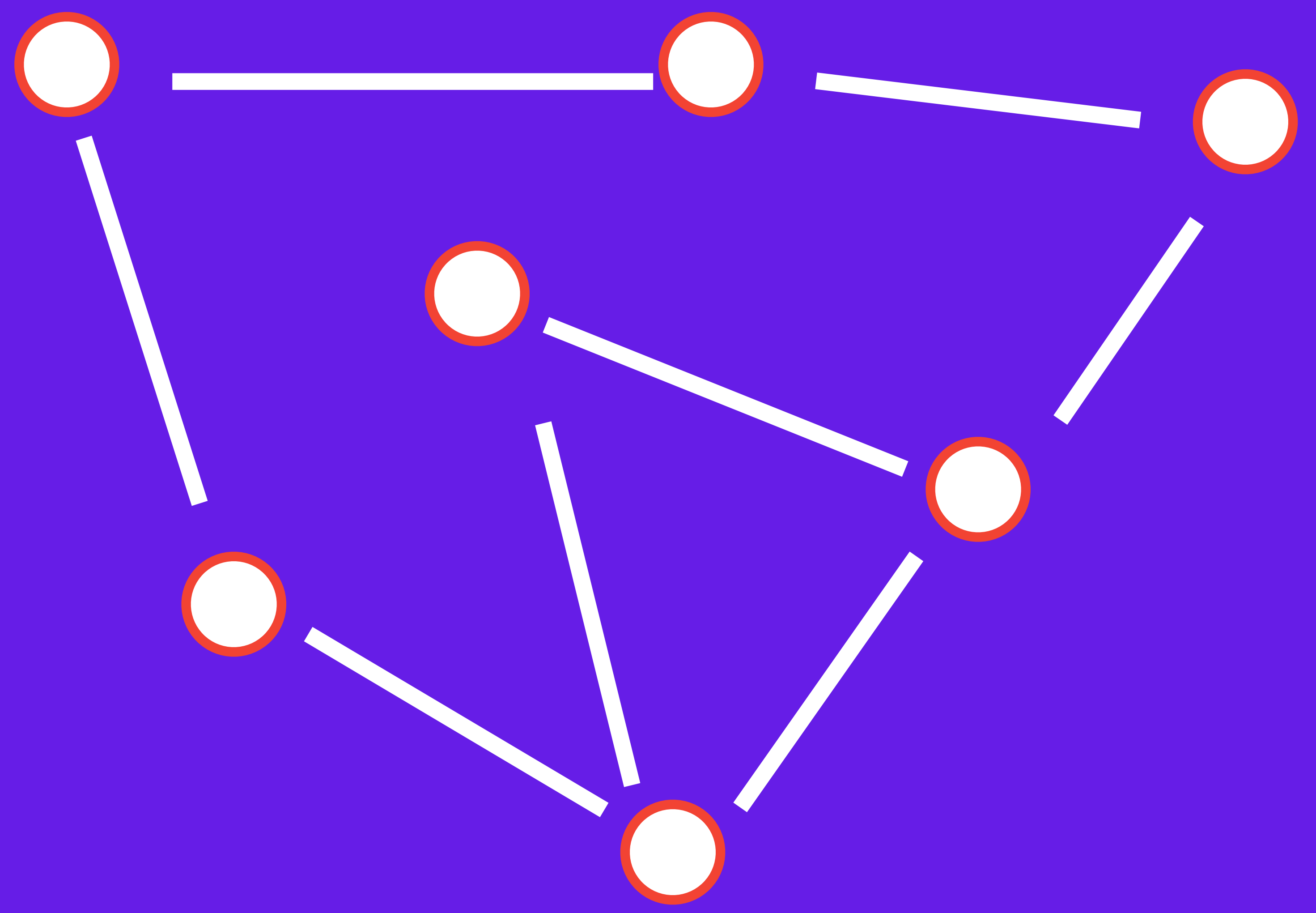
$$(x_1^1, x_2^1) = (0,1) \qquad 1x_1^1 + 8x_2^1 + 7x_2^2 + 4z_1 - 7z_2 \geq 8$$

$$(x_1^1, x_2^1) = (1,0) \qquad 1x_1^1 + 8x_2^1 - 4x_1^2 + 4z_1 - 7z_2 \geq 1$$

$$(x_1^2, x_2^2) = (1,0) \qquad 5x_1^1 + 10x_2^1 - 4x_1^2 + 5z_1 - 5z_2 \geq 5$$

$$(x_1^2, x_2^2) = (0,1) \qquad 5x_1^1 + 10x_2^1 + 7x_2^2 + 5z_1 - 5z_2 \geq 10$$

Network Formation Game



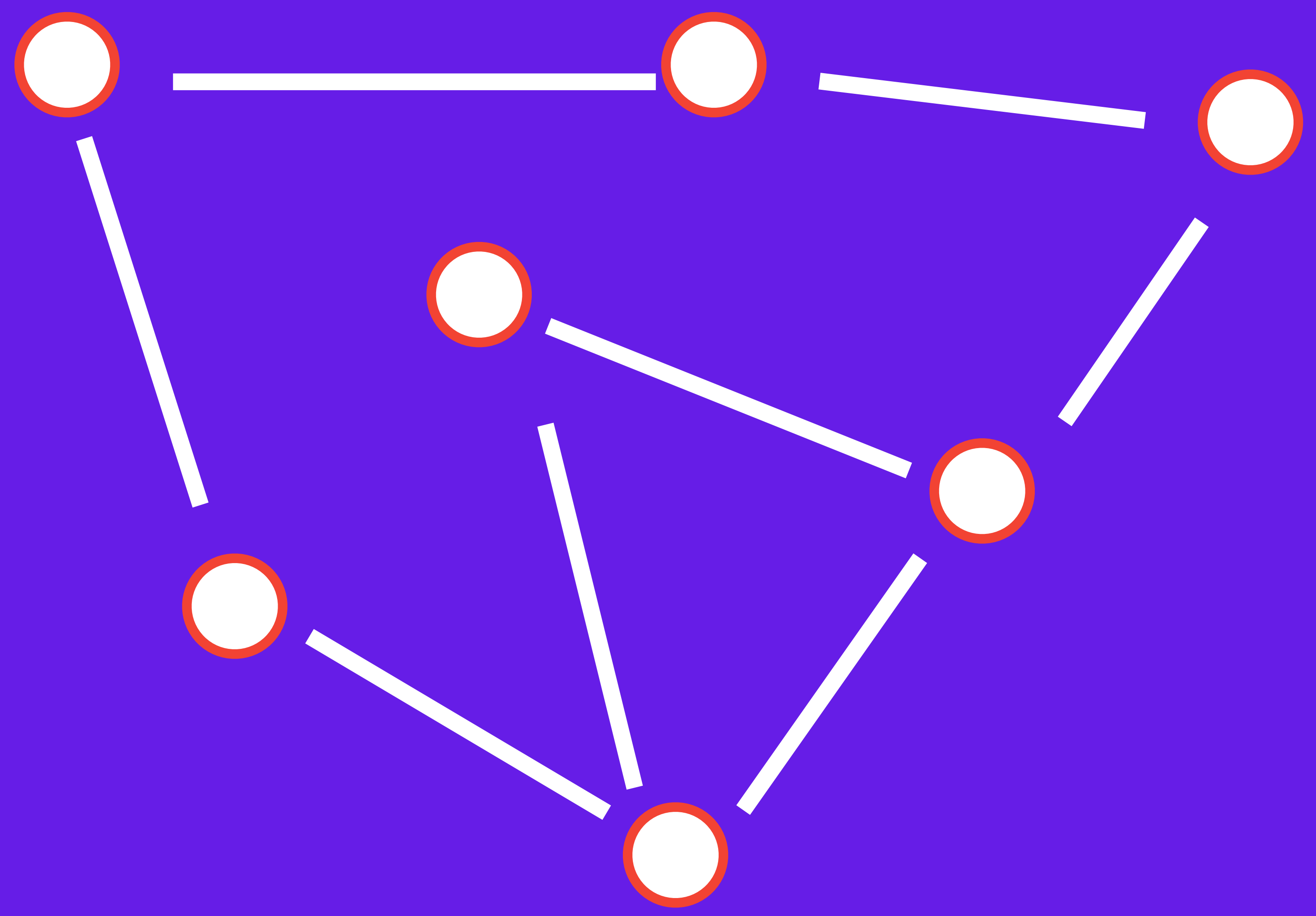
(Chen and Roughgarden, 2006; Anshelevich, et al., 2008; Nisan et al., 2008)

Given a graph $G = (V, E)$:

- Each edge $(h, l) \in E : h, l \in V$ has a cost $c_{hl} \in \mathbb{Z}^+$
- Each player i has a weight w^i to go from **a source** s^i **to a sink** t^i
- If more than one player selects an edge, the cost is **split proportionally to each player's weight** (Shapley sharing)

Selecting a PNE with $n \geq 3$ is $\mathcal{NP}$-hard

Network Formation Game



(Chen and Roughgarden, 2006; Anshelevich, et al., 2008; Nisan et al., 2008)

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- Each player i has a weight w^i to go from **a source** s^i **to a sink** t^i
- If more than one player selects an edge, the cost is **split proportionally to each player's weight** (Shapley sharing)

$$\min_{x^i} \{ \sum_{(h,l)\in E} \frac{c_{hl} x^i_{hl}}{\sum_{k=1}^{n} x^k_{hl}} : x^i \in \mathcal{F}^i \}$$

NASPs

The Problem(s)

Stackelberg Games

(Stackelberg, 1934; Candler and Norto, 1977)

A Stackelberg game is a **sequential game** with **perfect information** where the players act in **rounds:**

- We consider games where there is an **unique** first-round player called ***the leader,*** who solves an optimization problem
- The second-round players are ***the followers*** solving optimization problems **depending on the leader's choices**

A solution is a vector of strategies that are optimal for both the leader and its followers

In the general case, determining a solution is $\mathcal{NP}$-hard

Could we reformulate the Stackelberg game as a single optimization problem?

Not always, yet…

The Problem(s)

Stackelberg Games

(Basu et al., 2020)

A Stackelberg game **can be reformulated into a single-level optimization problem** if:

1. The leader's objective function is linear in its variables and the ones of its followers
2. The leader's constraints are linear constraints
3. The followers solve convex quadratic optimization problems

Specifically, the feasible region of this program is a **union of polyhedra**

Complexity

Complexity

NASPs

THEOREM

Given a NASP with 2 leaders with 1 follower each, so that each follower solves a linear program and the leaders all have linear objectives in their variables:

1. It is $\Sigma_2^p -$ hard to determine if the game an **MNE/PNE**
2. If all reformulated problems have a bounded feasible region $\mathscr{F}^i$, **there exists an MNE**

Algorithmic Ideas

Full Enumeration

INPUT: A NASP N

OUTPUT: a NE or none exists

```
For every player i = 1,2,…,n
  Compute cl conv(𝓕^i) through Balas'
```

```
Solve an LCP with the convex hulls
If LCP has a solution: return yes and NE
Else: return no NE exists
```

Inner approximation

Inner-approximate cl conv$(\mathscr{X}^i)$

Inner approximation

Compute an MNE starting with a single polyhedron

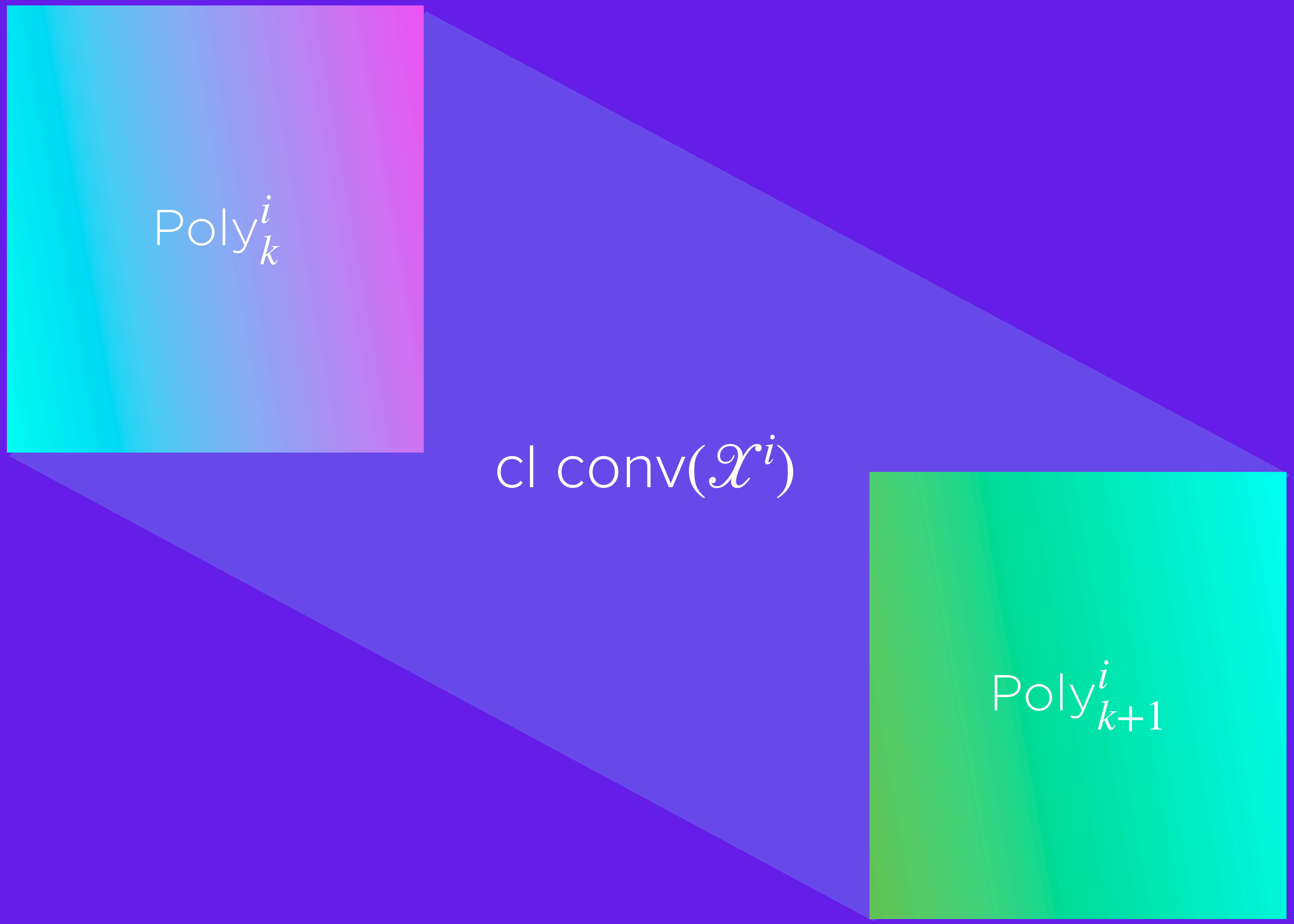
Inner approximation

If there **exists a NE,** and also a **deviation**, add it to the next iteration

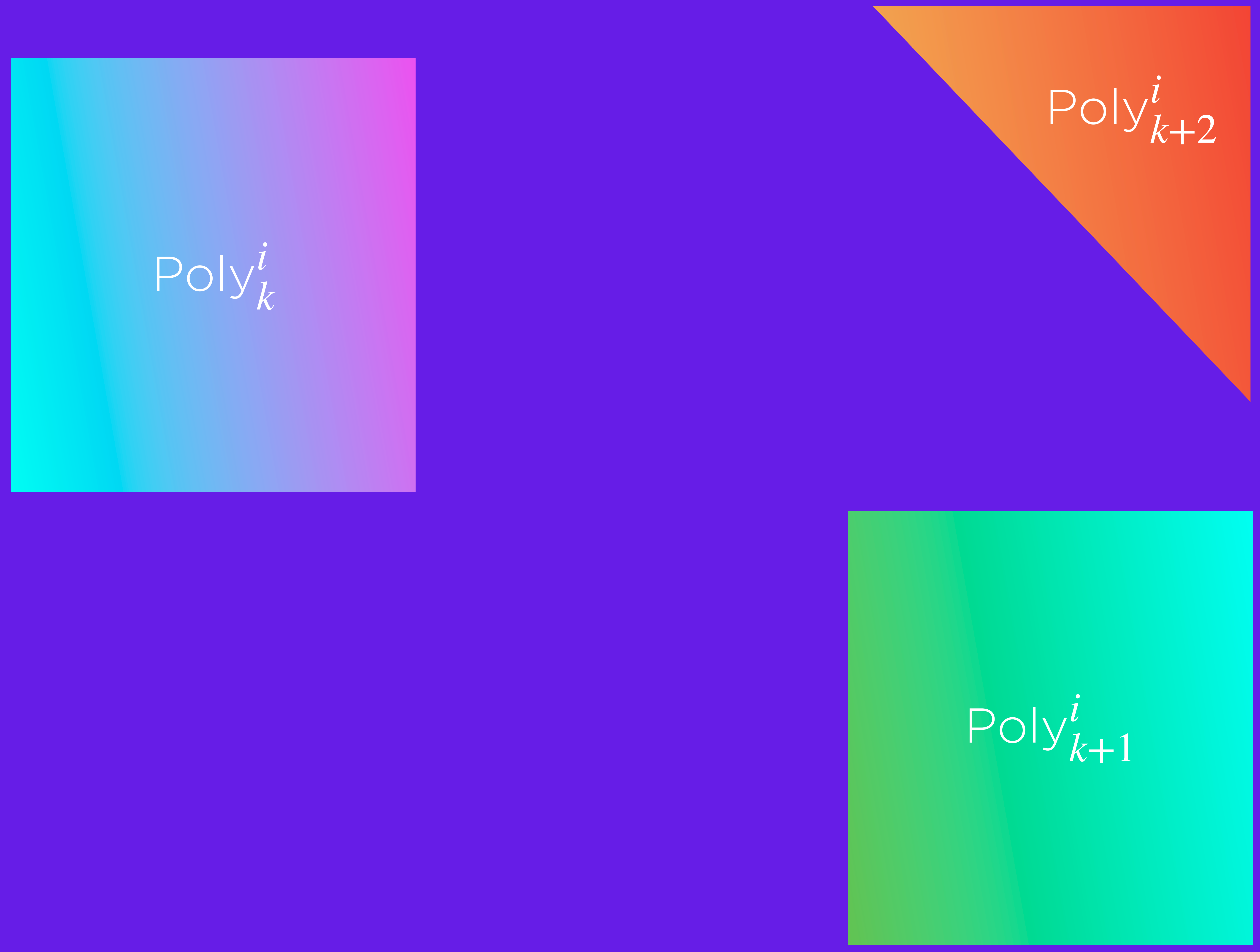
Poly_k^i

Poly_{k+1}^i

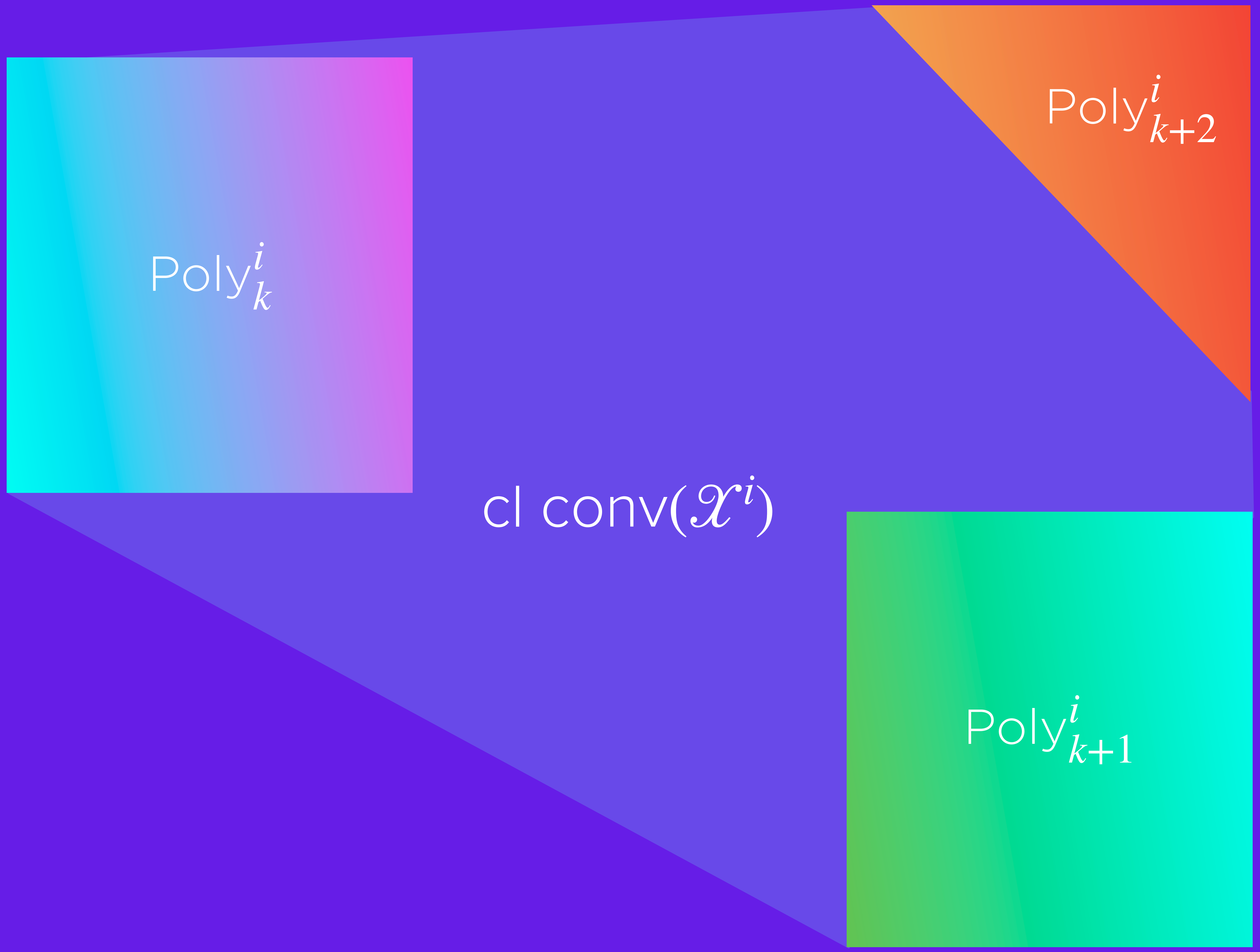
Inner approximation



Inner approximation



Inner approximation



Inner approximation

INPUT: A NASP N

OUTPUT: a NE or none exists

```
For every player i = 1,2,...,n
  Initialize F^i_* with one polyhedron from the union
While True:
  Solve an LCP to determine an NE
  If LCP has a solution:
      If no deviation: return yes and NE
      Else deviation for i: add the polyhedron to F^i_*
  If LCP has no solution:
      If no more polyhedra: return none exists
      Else: add random polyhedra to F^i_*
```

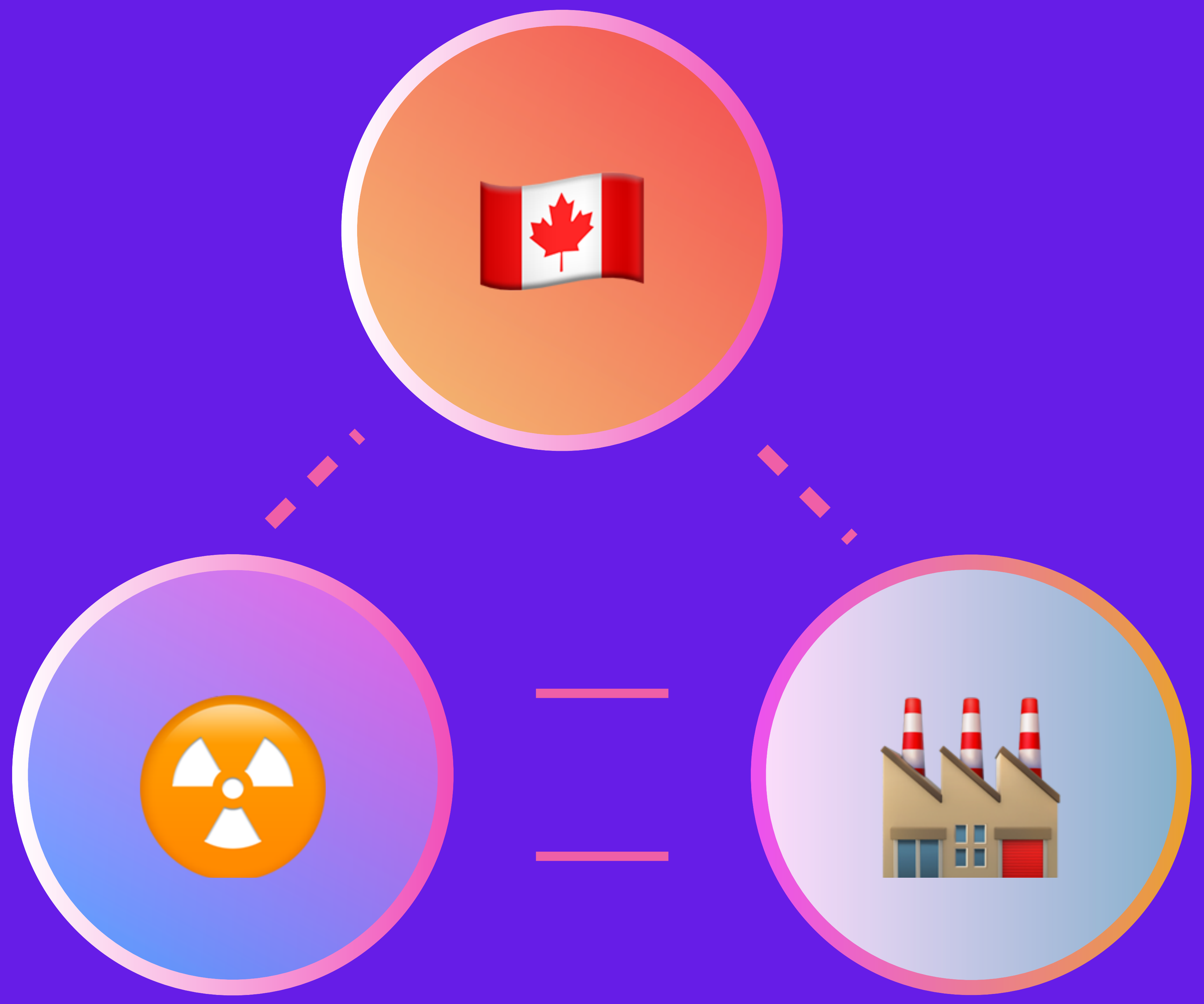
Insights

Magicville

The leader trades energy with other leaders

Followers produce energy and gets taxed by the leader

Magicville



The leader trades energy with other leaders

Followers produce energy and gets taxed by the leader

Are leaders (countries) further reducing their emission if they optimize the income from a carbon-tax?

Does trade among countries under a carbon-tax reduce emissions?

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It depends on what source energy producers use (i.e., coal vs solar).
In general, **no.**

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It depends on what source energy producers use (i.e., coal vs solar).
In general, **no.**

Does trade among countries under a carbon-tax reduce emissions?

Since trade is about money, the **intuitive answer is no.**
However, we found that countries with large quantities of clean energy can fulfil the need of countries with fossil fuel, **thus reducing the overall emissions.**

Clean Energy Experiments

Energy Game

	Algorithm	ES	k	Time (s) EQ	NO	All	Wins EQ	NO	Solved
	FE	-	-	29.08	0.12	120.21	6	82	140/149
MNE	*InnerApp*	Seq	1	6.65	0.35	51.33	3	0	145/149
		Seq	3	17.76	0.18	55.82	5	0	145/149
		Seq	5	6.40	0.15	51.08	3	0	145/149
		Rev.Seq	1	7.97	0.36	3.73	26	0	149/149
		Rev.Seq	3	11.29	0.18	53.12	4	0	145/149
		Rev.Seq	5	9.53	0.15	76.41	5	0	143/149
		Random	1	5.22	0.36	26.60	8	0	147/149
		Random	3	32.42	0.18	85.65	5	0	143/149
		Random	5	23.67	0.15	58.26	2	0	145/149
PNE	*FE-P*	-	-	7.25	0.12	328.23	–	–	122/149

Energy Game

Small

	Algorithm	ES	k	Time (s) EQ	NO	All	Wins EQ	NO	Solved
	FE	-	-	29.08	0.12	120.21	6	82	140/149
MNE	*InnerApp*	Seq	1	6.65	0.35	51.33	3	0	145/149
		Seq	3	17.76	0.18	55.82	5	0	145/149
		Seq	5	6.40	0.15	51.08	3	0	145/149
		Rev.Seq	1	7.97	0.36	3.73	26	0	149/149
		Rev.Seq	3	11.29	0.18	53.12	4	0	145/149
		Rev.Seq	5	9.53	0.15	76.41	5	0	143/149
		Random	1	5.22	0.36	26.60	8	0	147/149
		Random	3	32.42	0.18	85.65	5	0	143/149
		Random	5	23.67	0.15	58.26	2	0	145/149
PNE	*FE-P*	-	-	7.25	0.12	328.23	–	–	122/149

Energy Game

	Algorithm	ES	k	Time (s) EQ	NO	All	Wins EQ	NO	Solved
	FE	-	-	29.08	0.12	120.21	6	82	140/149
MNE	*InnerApp*	Seq	1	6.65	0.35	51.33	3	0	145/149
		Seq	3	17.76	0.18	55.82	5	0	145/149
		Seq	5	6.40	0.15	51.08	3	0	145/149
		Rev.Seq	1	7.97	0.36	3.73	26	0	149/149
		Rev.Seq	3	11.29	0.18	53.12	4	0	145/149
		Rev.Seq	5	9.53	0.15	76.41	5	0	143/149
		Random	1	5.22	0.36	26.60	8	0	147/149
		Random	3	32.42	0.18	85.65	5	0	143/149
		Random	5	23.67	0.15	58.26	2	0	145/149
PNE	*FE-P*	-	-	7.25	0.12	328.23	–	–	122/149

Energy Game

	Algorithm	ES	k	Time (s) EQ	NO	All	Wins EQ	NO	Solved
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		Rev.Seq	3	11.29	0.18	53.12	4	0	145/149
		Rev.Seq	5	9.53	0.15	76.41	5	0	143/149
		Random	1	5.22	0.36	26.60	8	0	147/149
		Random	3	32.42	0.18	85.65	5	0	143/149
		Random	5	23.67	0.15	58.26	2	0	145/149
PNE	*FE-P*	-	-	7.25	0.12	328.23	–	–	122/149

Energy Game

Large

	Algorithm	ES	k	Time (s) EQ	NO	All	Wins EQ	NO	Solved
	FE	-	-	260.29	1.12	1174.32	0	2	20/50
MNE	*InnerApp*	Seq	1	39.26	9.64	672.24	1	0	32/50
		Seq	3	62.66	3.88	616.25	1	0	34/50
		Seq	5	24.03	2.83	733.97	1	0	30/50
		Rev.Seq	1	171.47	9.66	262.74	27	0	47/50
		Rev.Seq	3	13.85	3.86	585.27	4	0	34/50
		Rev.Seq	5	78.57	2.83	798.90	6	0	29/50
		Random	1	34.65	9.65	497.06	0	0	37/50
		Random	3	123.02	3.87	588.03	2	0	36/50
		Random	5	39.18	2.86	711.77	4	0	41/50
PNE	*FE-P*	-	-	7.36	1.12	1441.95	–	–	10/50

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PNE	*FE-P*	-	-	7.36	1.12	1441.95	–	–	10/50

NASPs

Algo	Inst	#	GT (s)	# NASH_EQ	GT (s)	# NO_EQ	GT (s)	#N ALL	#NI	#TL
Inn-S-1	**B**	50	6.22	49	69.76	1	6.56	50	0	0
Inn-S-3	**B**	50	4.94	49	23.96	1	5.12	50	0	0
Out-HB	**B**	50	7.47	46	29.37	1	7.71	47	3	0
Out-DB	**B**	50	9.45	46	11.81	1	9.50	47	3	0
Inn-S-1	**H7**	50	-	0	-	0	300.00	46	4	46
Inn-S-3	**H7**	50	-	0	-	0	-	0	50	0
Out-HB	**H7**	50	53.79	41	-	0	73.45	50	0	9
Out-DB	**H7**	50	52.58	35	-	0	88.92	50	0	15