Integer Programming Games

What, Why and How?

Gabriele Dragotto

Politecnico di Torino
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Accelerating Scientific Discovery at Princeton

Politecnico di Torino
1859

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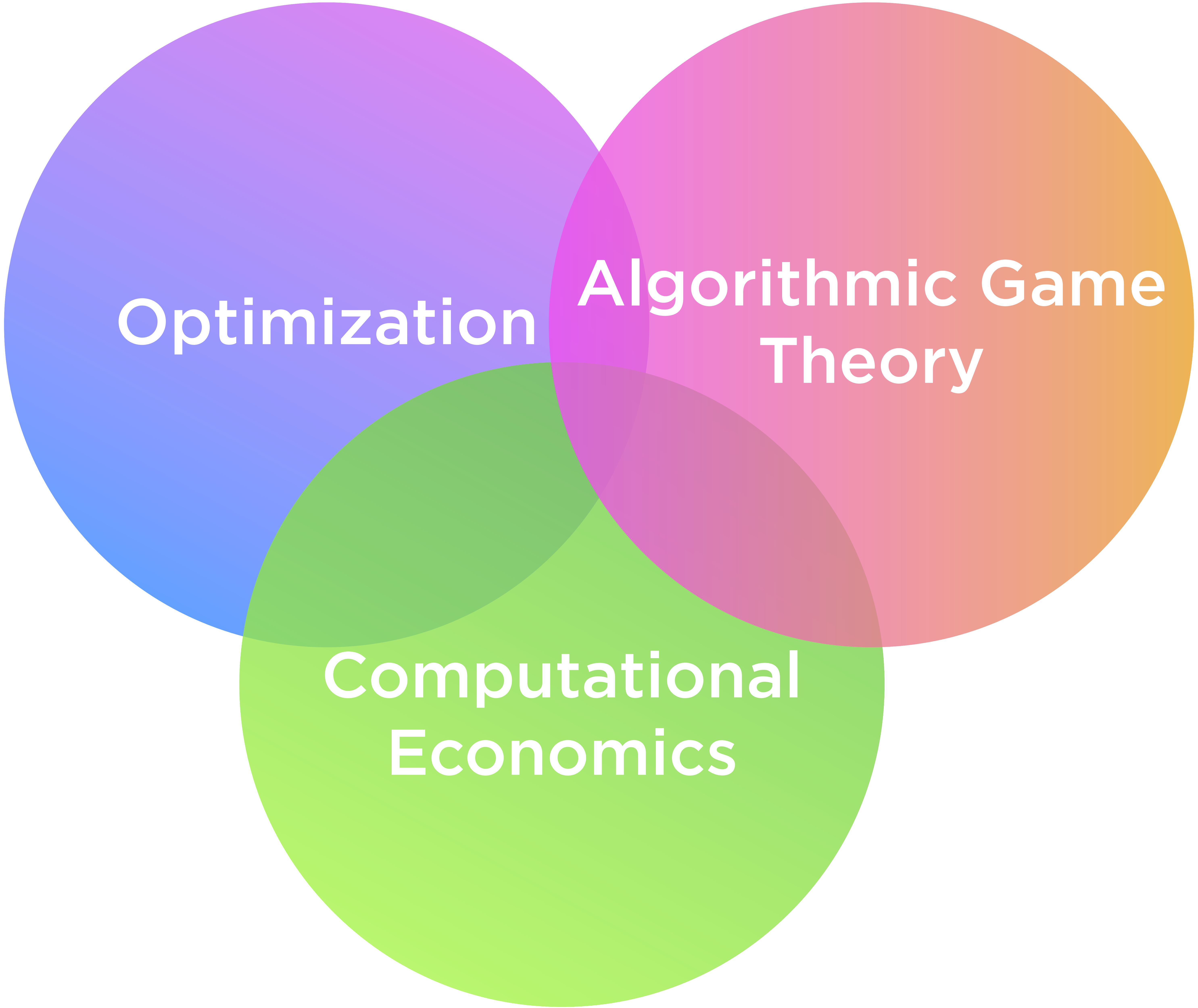
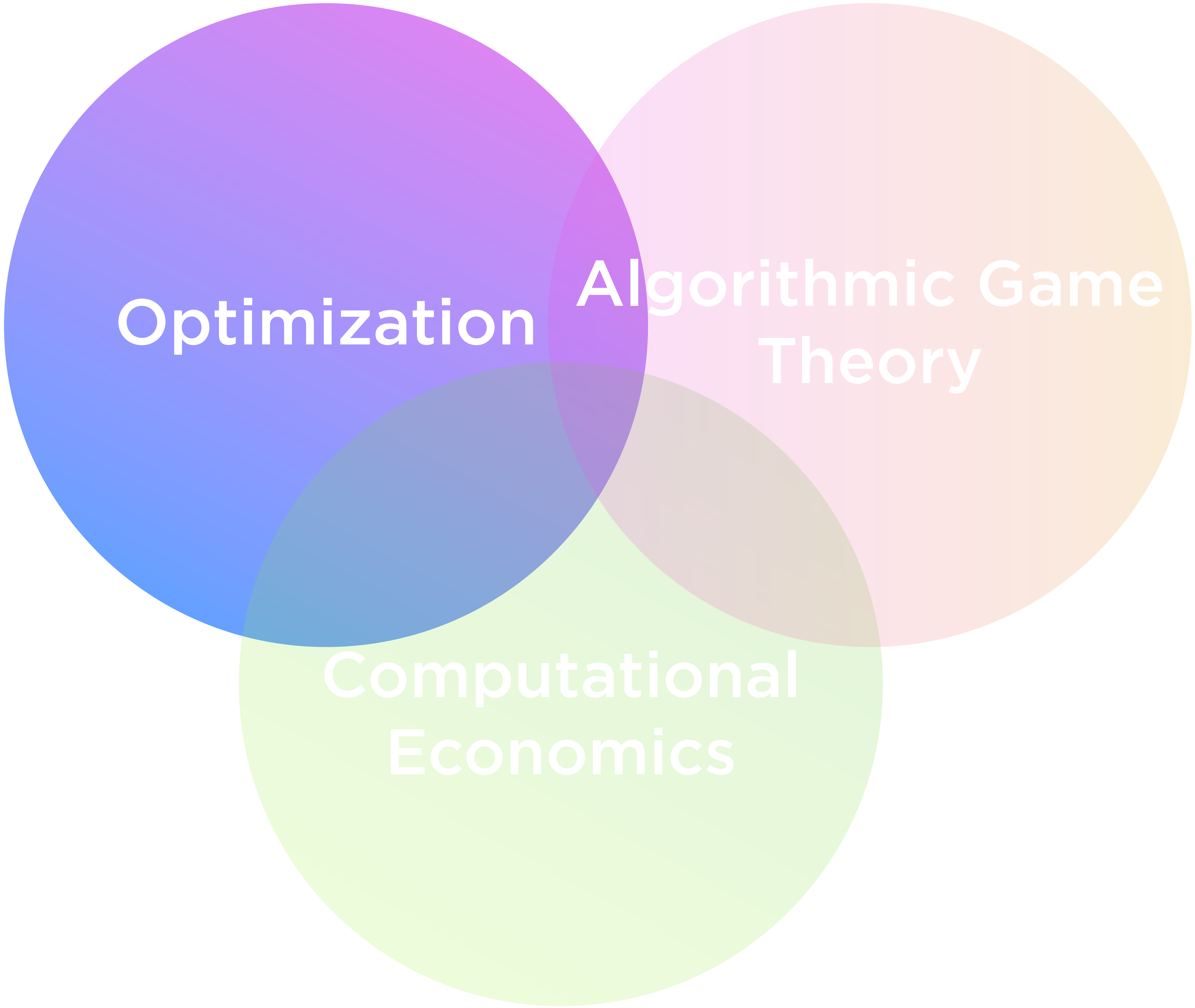

Optimization
Algorithmic Game Theory
Computational Economics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Politecnico di Torino

Andrea Lodi
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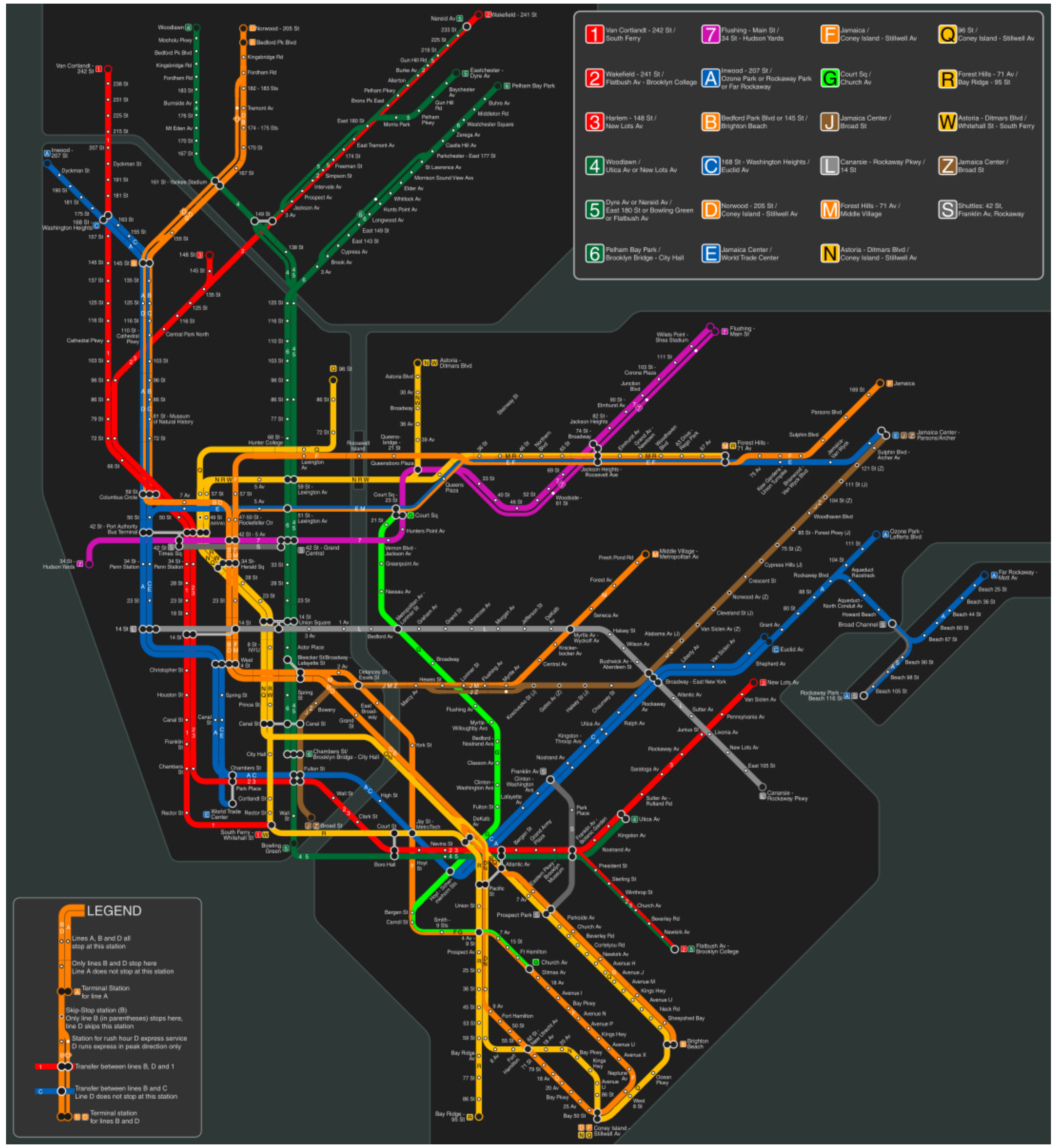
What?

Commuting

Network Formation



There are n players optimizing simultaneously the **shortest path** on a network, and want to **share** the **setup costs**

Choices of **other players**

$$\min_{x^i} \{ u^i (x^i ; x^{-i}) : x^i \in \mathcal{X}^i \}$$

Choices of **player** i

How do we **algorithmically compute** the **best stable outcome?**

Network Congestion

A regulator wants to **intervene in the game**

Multi-agent Assortment

$$
\begin{aligned}
\max_{x^1} \quad & 6x_1^1 + x_2^1 \\
\text{s.t.} \quad & 3x_1^1 + 2x_2^1 \leq 4 \\
& x^1 \in \{0,1\}^2
\end{aligned}
$$

Their “profits” **interact**

$$\begin{aligned} \max_{x^1} \quad & 6x^1_1 + x^1_2 - 4x^1_1x^2_1 + 6x^1_2x^1_2 \\ \text{s.t.} \quad & 3x^1_1 + 2x^1_2 \leq 4 \\ & x^1 \in \{0,1\}^2 \end{aligned}$$

$$\begin{aligned} \max_{x^2} \quad & 4x^2_1 + 2x^2_2 - x^2_1x^1_1 - x^2_2x^1_2 \\ \text{s.t.} \quad & 2x^2_1 + 3x^2_2 \leq 4 \\ & x^2 \in \{0,1\}^2 \end{aligned}$$

Knapsack Games (Carvalho et al., 2022; ***D.*** *and Scatamacchia, 2022), Critical Node Game (**D.** et al., 2023)*

Energy Markets

SolarCorp Inc.

Simultaneous Game

Hydro Inc.

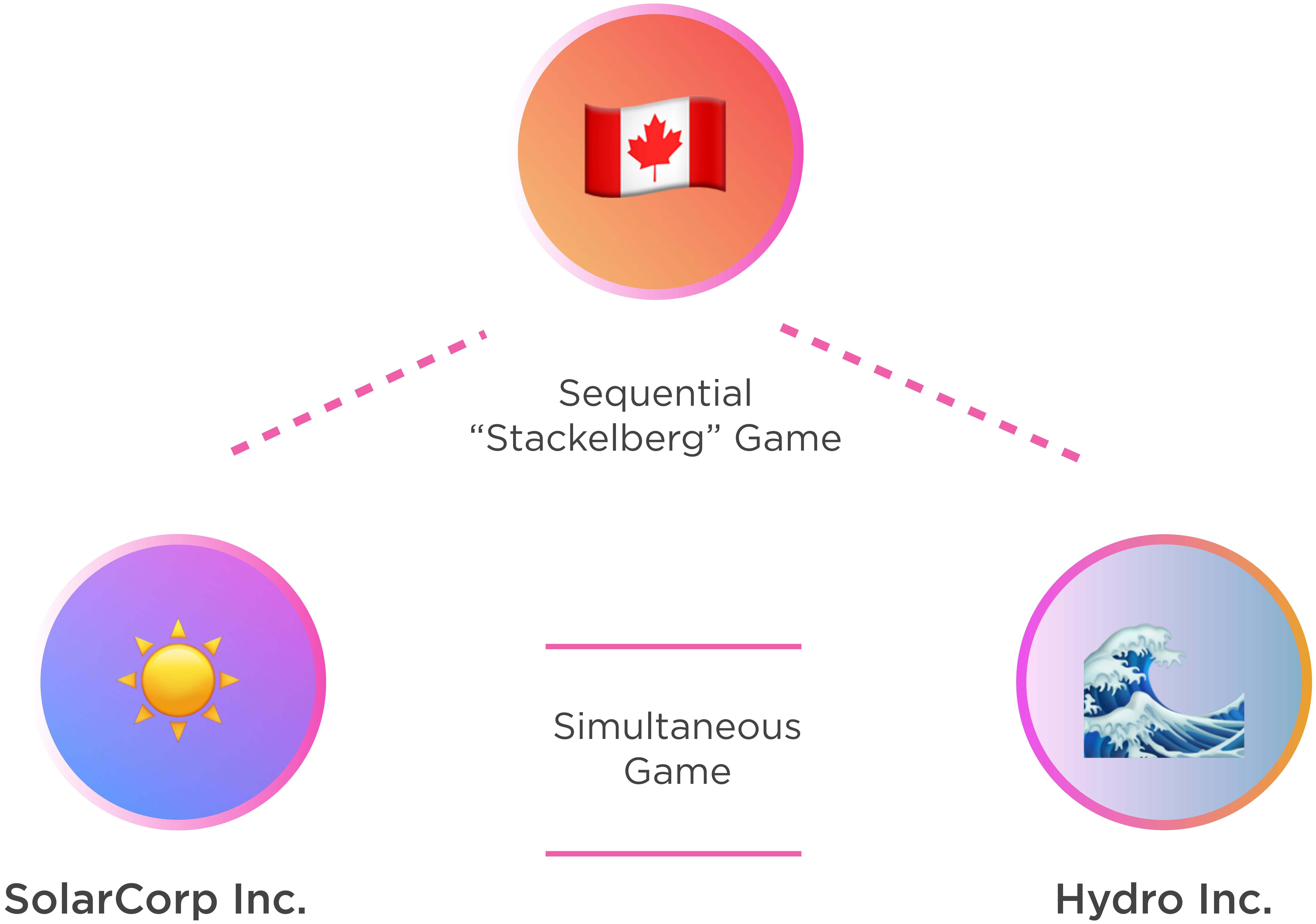
“Cournot Game”

Canada taxes and regulates the production

SolarCorp Inc.

Simultaneous Game

Hydro Inc.


Sequential
“Stackelberg” Game
Simultaneous
Game
SolarCorp Inc.
Hydro Inc.

This is a simultaneous game among **optimistic bilevel** (i.e., sequential) programs

When Nash Meets Stackelberg (Carvalho et al., 2022)

And it can get more complex…

And it get more complex…

Supply Chain and Transportation

Cronert and Minner, 2021 (OR, TR-B)

Sagratella et al 2020 (EJOR)

Carvalho et al. 2018 (IJ Production Economics)

Simultaneous game among "bilevel" players

*Carvalho, **D.** et al, 2023*

(Management Science)

Cybersecurity

***D.** et al, 2023*

(Ericsson Inc, - Patent pending)

Decision-making is rarely an individual task

Self-driven interactions with other decision-makers

Decision-making is rarely an individual task

Self-driven interactions with other decision-makers

deciding by solving **complex (e.g., non-convex) optimization problems**

- Modeling Capabilities
- Informative Solutions
- Practically Useful

The Toolkit: Integer Programming Games

An Integer Programming Game (IPG) is a *simultaneous one-shot (static)* game among n players where each player $i = 1, \ldots, n$ solves

$$\min_{x^i} \{u^i(x^i; x^{-i}) : x^i \in \mathcal{X}^i\}$$

$$\mathcal{X}^i := \{A^i x^i \leq b^i, \;\; x^i \in \mathbb{Z}^{\alpha^i} \times \mathbb{R}^{\beta^i}\}$$

There is **common knowledge of rationality**, i.e., each player is **rational** and there is **complete information**

*Köppe et al., (2011), Sagratella (2015), **D.** et al (2021), **D.** (2022)*

Why?

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Decision-making is rarely an individual task

Self-driven interactions with other decision-makers

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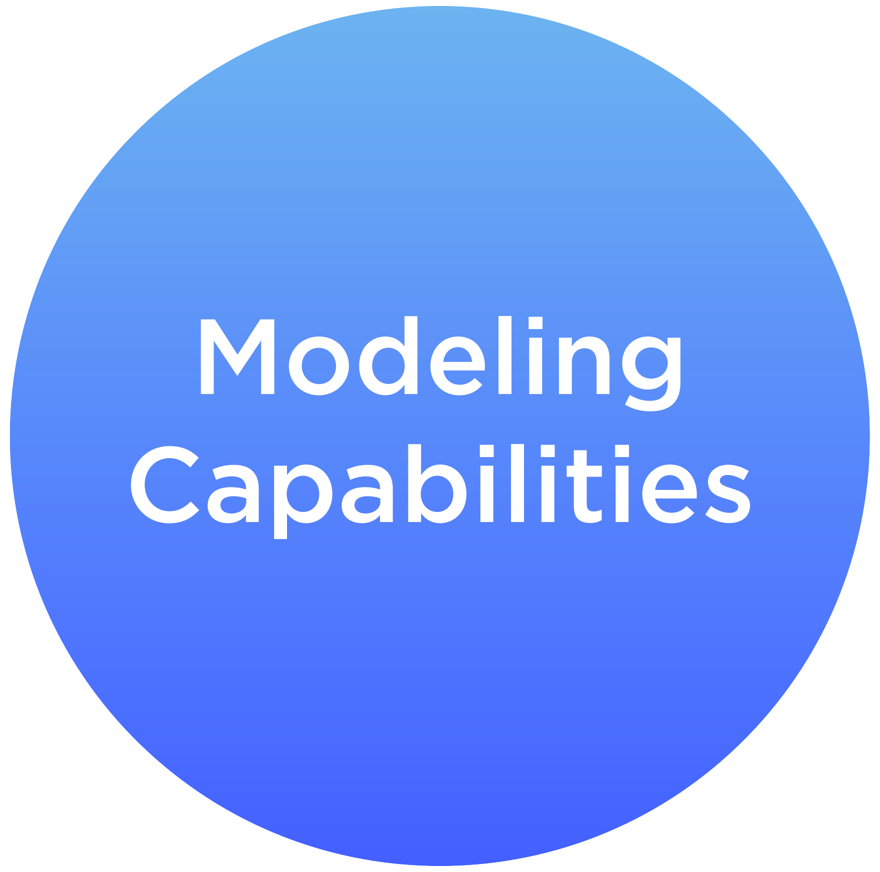
Modeling Capabilities

Informative Solutions

Practically Useful

Why? Modeling Capabilities



They extend traditional **resource-allocation tasks and combinatorial optimization** problems to a multi-agent setting

Indivisible quantities, **fixed production costs** and **logical disjunctions** often require discrete variables

Energy — Gabriel et al. (2013), David Fuller and Çelebi (2017)
Supply Chain — Anderson et al. (2017)
Assortment-Price competitions — Federgruen and Hu (2015)
Kidney Exchange Problems — Carvalho et al. (2017)
Cybersecurity — Dragotto et al. (2023)

Why? Informative Contents of Equilibria

Informative Solutions

Econometrica, Vol. 70, No. 4 (July, 2002), 1341–1378

THE ECONOMIST AS ENGINEER: GAME THEORY, EXPERIMENTATION, AND COMPUTATION AS TOOLS FOR DESIGN ECONOMICS[1]

BY ALVIN E. ROTH[2]

*"Designers therefore **cannot work** only **with the simple conceptual models** used for theoretical insights into the general working of markets. Instead, **market design calls for an engineering approach**.*

***Experimental and computational economics** are natural complements to game theory in the work of design."*

How?

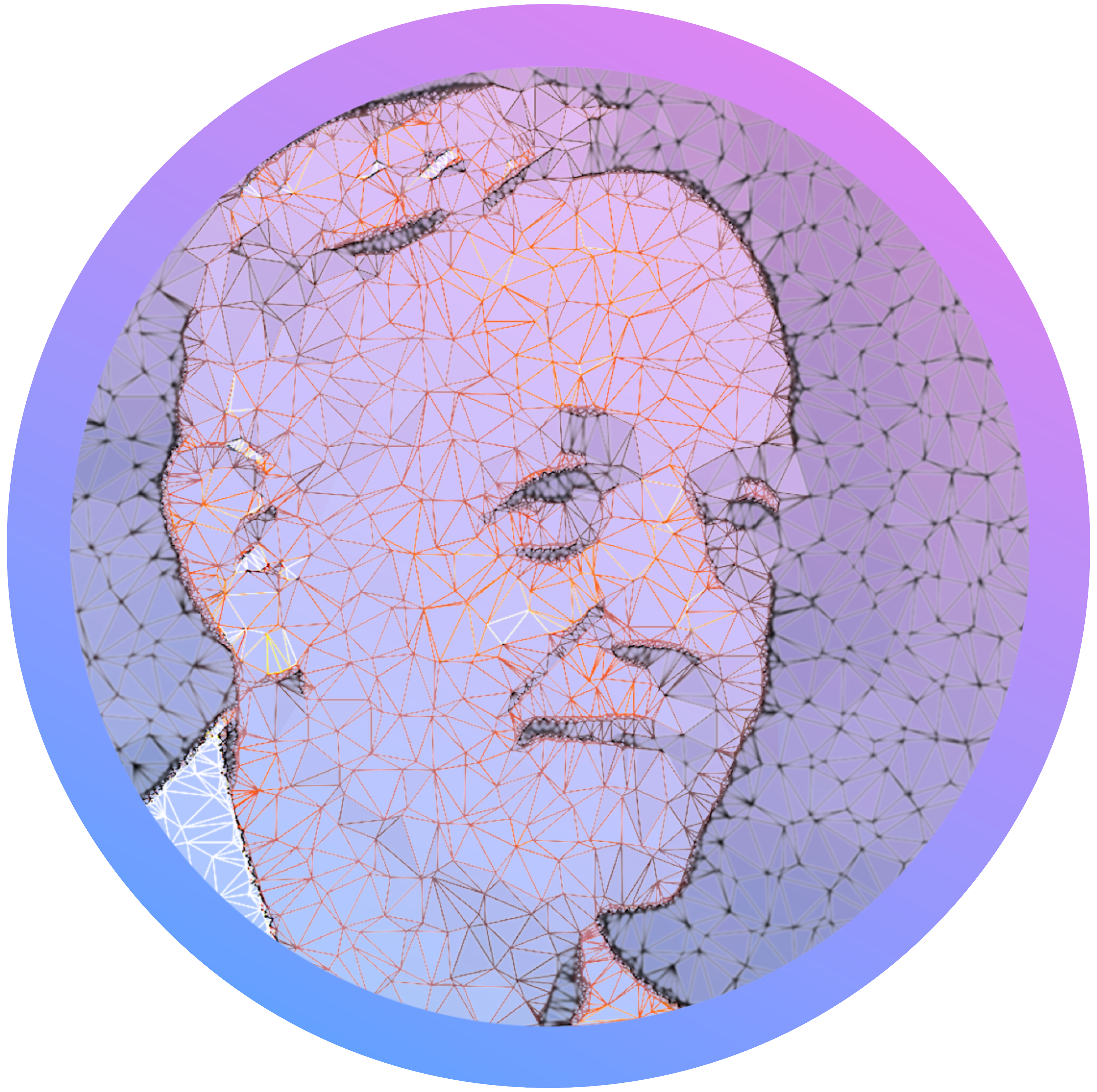
Decision-making is rarely an individual task

Self-driven interactions with other decision-makers

Solutions should embed the concept of ***stability***

For instance, **the mutual optimality** of the decision-makers

Nash equilibria



$\bar{x} = (\bar{x}^1, \ldots, \bar{x}^n)$ *is a Pure Nash Equilibrium (***PNE***) if, for any player* i,

$$u^i(\bar{x}^i, \bar{x}^{-i}) \leq u^i(\hat{x}^i, \bar{x}^{-i}) \quad \forall \hat{x}^i \in \mathcal{X}^i$$

PNEs and MNEs *(Carvalho et. al, 2018)*

1. Deciding if an IPG has a PNE is Σ_2^p-complete
2. Deciding if an IPG has a MNE is Σ_2^p-complete
3. If $\mathcal{X}^i$ is finite for any player i, there exists an MNE

State-of-the-art

		Enumerate	Optimize	Payoff Types
Sagratella (2016)	Branching Method	✅	❌	Convex payoffs
Carvalho et al. (2022)	Sample Generation Method	❌	❌	Separable Payoffs
Schwarze and Stein (2022)	Branch-and-Prune	✅	❌	Quadratic Payoffs
Carvalho, D., Lodi, Sankaranarayanan (2021)	Cut-And-Play	❌	❌	Separable Payoffs
Cronert and Minner (2021)	Exhaustive Sample Generation Method	✅	❌	Separable Payoffs
D. and Scatamacchia (2023)	Zero Regrets	✅	✅	Linearizable Payoffs

A 3-Phase Process

INPUT A game G described by the payoff $u^i(x^i; x^{-i})$ and the actions $\mathcal{X}^i$

1. **Approximate** — Create $\tilde{G}$: Approximate $\mathcal{X}^i$ with some set $\tilde{\mathcal{X}}^i$
2. **Play** — Compute a solution $\bar{x}$ to $\tilde{G}$ with an optimization problem
3. **Refine** — Check if $\bar{x}$ is a Nash equilibrium.

   If not, refine at least a $\mathcal{X}^i$ and goto 2

   Else: $\bar{x}$ **is a Nash equilibrium**

A 3-Phase Process

3 **Refine**

Check if $\bar{x}$ is a Nash equilibrium.

If not, refine at least a $\mathcal{X}^i$ and goto 2

Else: $\bar{x}$ **is a Nash equilibrium**

Stability Step

No player should deviate from $\bar{x}^i$ given $\bar{x}^{-i}$

$$\tilde{x}^i = \arg\min_{x^i}\{u_i(x^i, \bar{x}^{-i}) : x^i \in \mathcal{X}^i\}$$

$$u^i(\bar{x}^i; \bar{x}^{-i}) \leq u^i(\tilde{x}^i; \bar{x}^{-i})$$

Membership Step

If $\bar{x}^i \notin \mathcal{X}^i$, then refine $\tilde{\mathcal{X}}^i$

And some challenges

- Existence of equilibria
- How to select approximations
- Solving the play phase
- Equilibria Selection

And some challenges

Existence of equilibria

How to select approximations

Solving the play phase

Equilibria Selection

An equilibrium **might not exist** in a given game G.
How to detect this?

And some challenges

Existence of equilibria

How to select approximations

Solving the play phase

Equilibria Selection

An equilibrium **might not exist** in an approximation $\tilde{G}$ while one exists in the original game G (and vice versa)

Outer vs Inner approximations

And some challenges

Existence of equilibria

How to select approximations

Solving the play phase

Equilibria Selection

This phase often involves solving a **hard optimization problem**,
e.g., a mixed-integer program, variational inequality or complementarity problem.

And some challenges

- Existence of equilibria
- How to select approximations
- Solving the play phase
- Equilibria Selection

If multiple equilibria exist, **selecting (i.e., optimizing) a specific one** is often complex from an algorithmic-design perspective

An example: Cut-and-Play

Cut-and-Play

INPUT A game G described by the **separable** payoff $u^i(x^i; x^{-i})$ and the actions $\mathcal{X}^i$

1 **Approximate**

Create $\tilde{G}$: Approximate $\mathcal{X}^i$ with some **polyhedron** $\tilde{\mathcal{X}}^i$

2 **Play**

Compute a solution $\bar{x}$ to $\tilde{G}$ via a **complementarity problem**

3 **Refine**

Check if $\bar{x}$ is a Nash equilibrium via both membership and stability

If not, refine at an $\mathcal{X}^i$ via **branching/cutting** 2

Else: $\bar{x}$ **is a Nash equilibrium**

Looking Ahead

Some perspectives

Applications

Deployment of IPGs in **novel application domains**

Optimization

Develop **novel and efficient** general and problem-specific algorithms

Fairness

Solutions **balancing** the decision-makers selfishness with societal goals

Integer Programming Games: A Gentle Computational Overview

INFORMS 2023 TutORial in O.R. - 2023

arXiv 2303.11188

The Cut-and-Play Algorithm

arXiv 2111.05726

DataX

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Knapsack Game (*KPG*)

As for Wizard and Fairy, each player solves a binary Knapsack problem with some **interaction terms** in the objective

$$\max_{x^i} \left\{ \sum_{j=1}^{m} p_j^i x_j^i + \sum_{k=1, k \neq i}^{n} \sum_{j=1}^{m} C_{k,j}^i x_j^i x_j^k : \sum_{j=1}^{m} w_j^i x_j^i \leq b^i, \mathbf{x}^i \in \{0,1\}^m \right\}$$

Knapsack Game (*KPG*)

A few facts:

- No successful attempts to **enumerate or select** equilibria in KPGs with $n > 2$ and $m > 4$ (*Cronert and Minner (2021)*)
- Carvalho et al. (2021, 2022) only compute **an MNE** with at most $n = 3$, $m \leq 40$
- No results on the complexity of the KPG, nor its PoS/PoA

We select PNEs with $n > 2$, $m > 50$
We provide "packing" equilibrium inequalities

We prove it is Σ_2^p-complete to determine if a PNE exists + the PoS/PoA are arbitrarily bad

Knapsack Game (*KPG*)

Equilibrium inequalities may also **capture specific structures** or constraint types.

Strategic Payoff Inequalities

A fact

In a packing problem, often the all-zeros strategy is feasible with objective 0

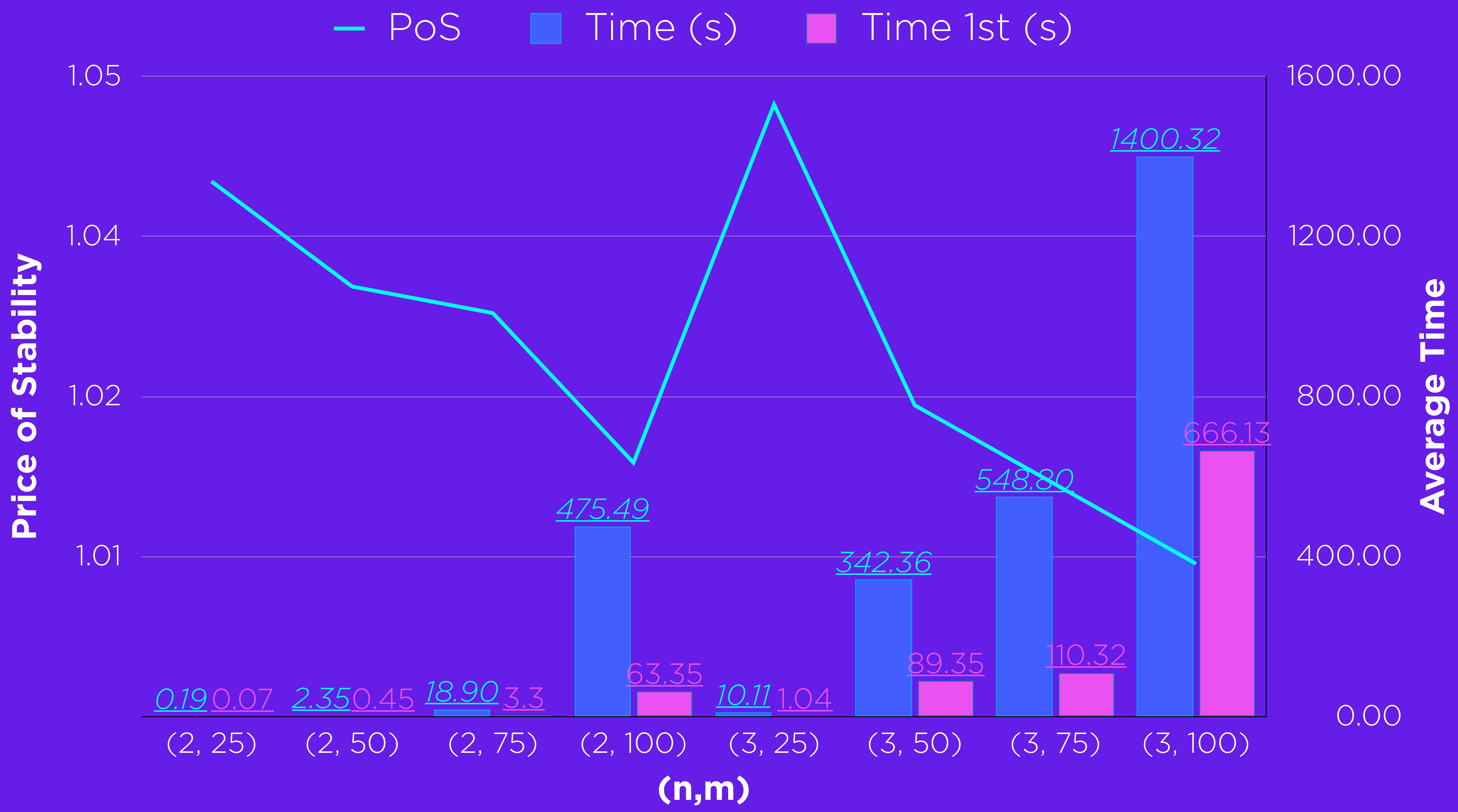
A consequence

Let $\mathcal{S}_i$ be a subset of i's opponents. If $\exists \mathcal{S}_i$ so that

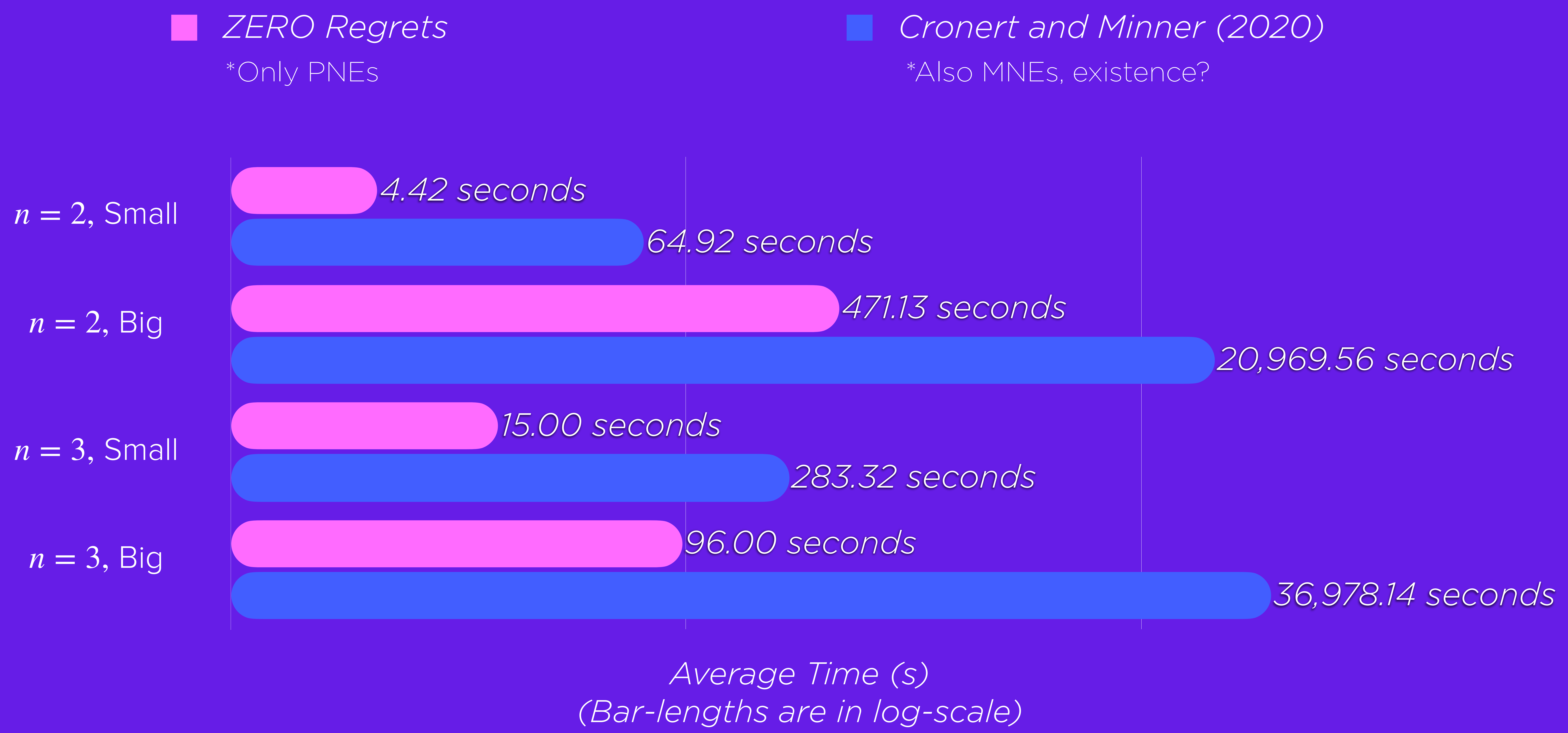
$$p_j^i + \sum_{k \in \mathcal{S}_j^i} C_{k,j}^i < 0,$$

then, $x_j^i + \sum_{k \in \mathcal{S}_j^i} x_j^k \leq |\mathcal{S}_j^i|$ is an **equilibrium inequality**.

Knapsack Game



Facility Location and Design Game



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