

Integer Programming Games

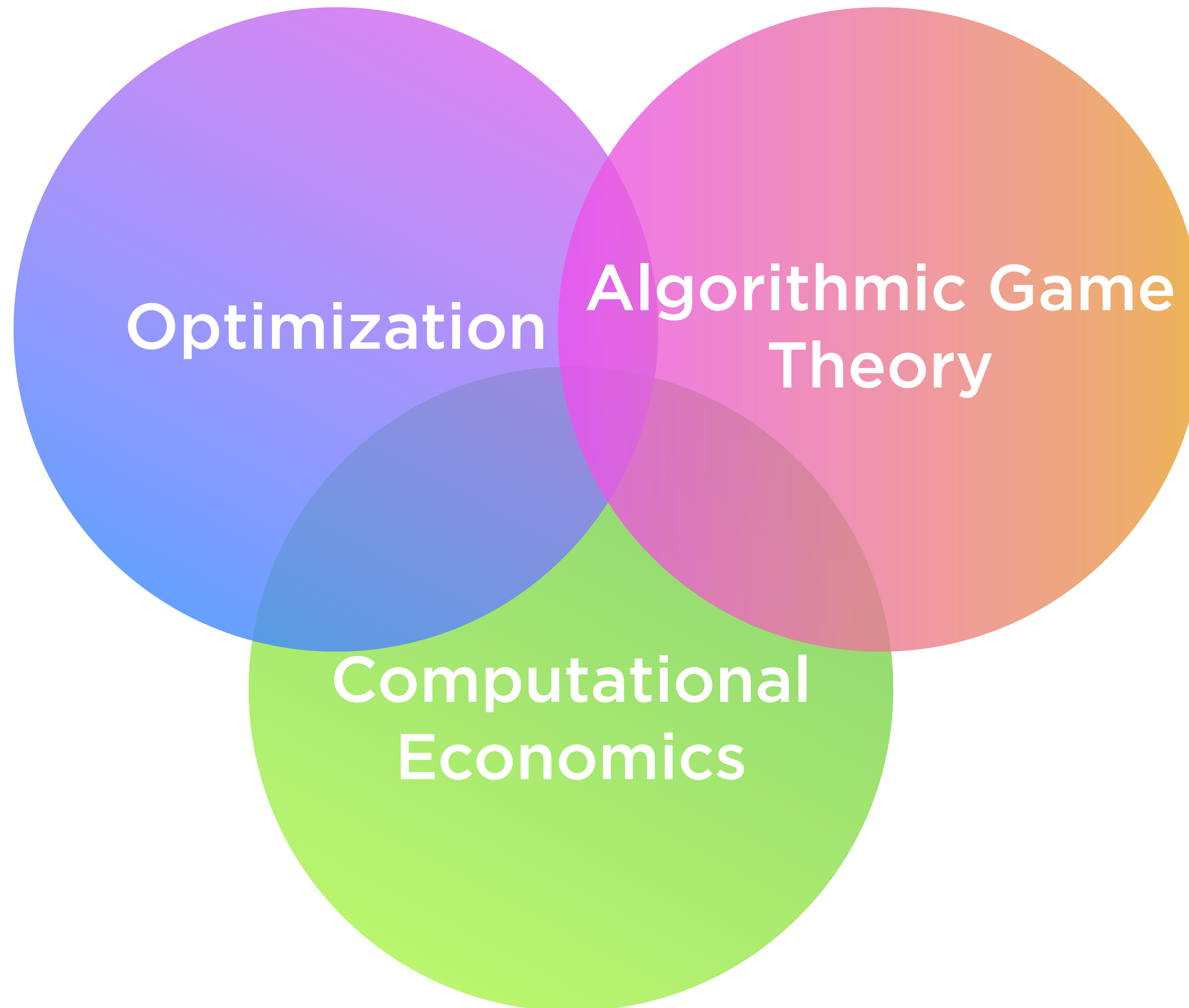
A Gentle Computational Overview

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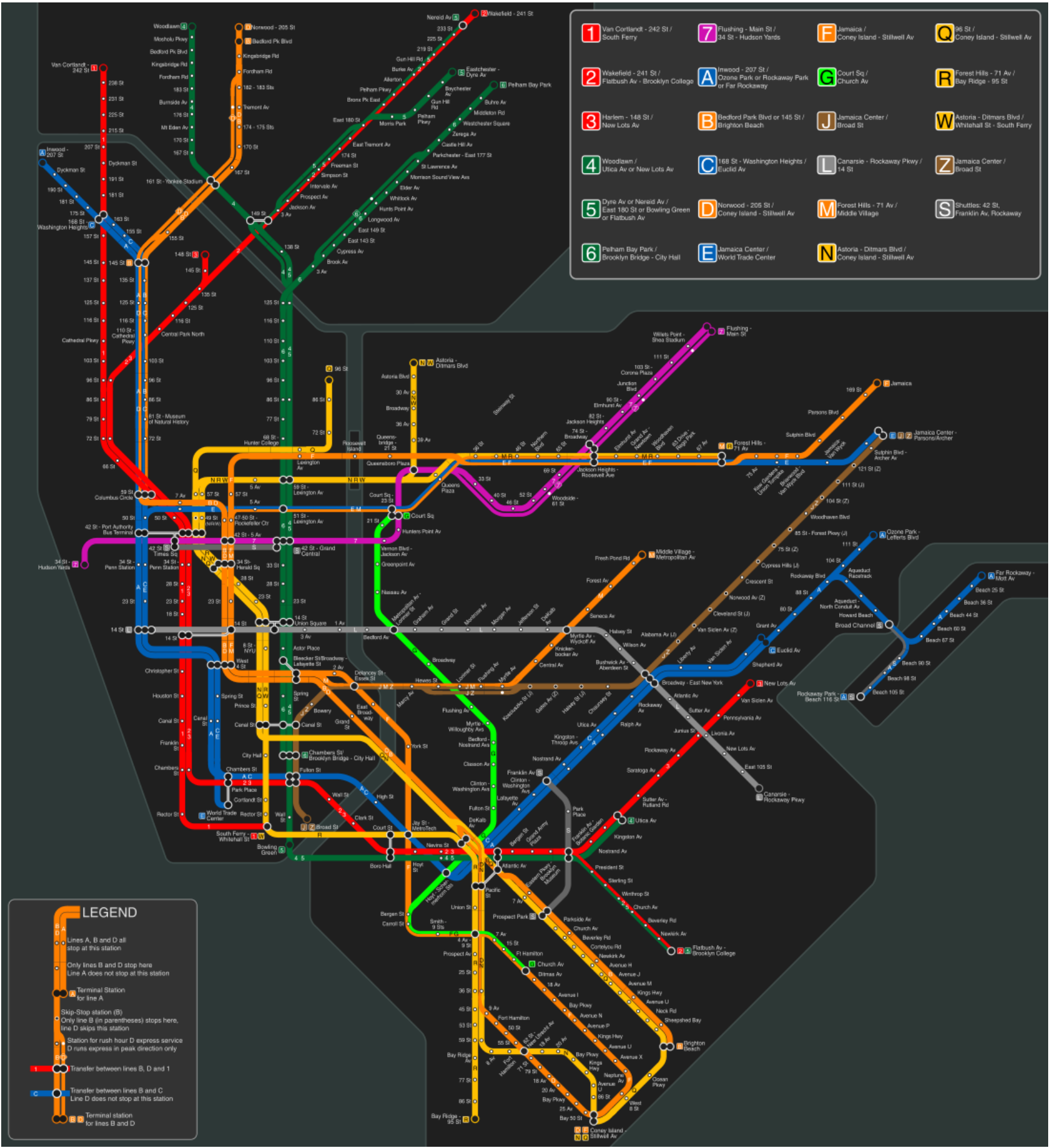


What are IPGs?



Commuting

Congestion Games



There are n players simultaneously optimizing the **shortest path** on a network

Choices of **other players**

$$\min_{x^i} \{ u^i(x^i; x^{-i}) : x^i \in \mathcal{X}^i \}$$

Choices of **player i**

How do we **algorithmically compute** a *stable outcome*?



Multi-agent Assortment



$$\begin{aligned} &\underset{x^1}{\text{maximize}} && x_1^1 + 2x_2^1 \\ &\text{s.t.} && 3x_1^1 + 4x_2^1 \leq 5, \\ &&& x^1 \in \{0, 1\}^2 \end{aligned}$$



$$\begin{aligned} &\text{maximize}_{x^1} \quad \text{🍏} + 2 \text{🧀} \\ &\text{s.t.} \quad 3 \text{🍏} + 4 \text{🧀} \leq 5, \\ &\quad \quad x^1 \in \{0, 1\}^2 \end{aligned}$$



Their “profits” **interact**



$$\begin{aligned}
 &\text{maximize}_{x^1} \quad \text{🍏} + 2 \text{🧀} \\
 &\text{s.t.} \quad 3 \text{🍏} + 4 \text{🧀} \leq 5, \\
 &\quad x^1 \in \{0, 1\}^2
 \end{aligned}$$

$$\begin{aligned}
 &\text{maximize}_{x^2} \quad 3 \text{🍏} + 5 \text{🍫} \\
 &\text{s.t.} \quad 2 \text{🍏} + 5 \text{🍫} \leq 5, \\
 &\quad x^2 \in \{0, 1\}^2
 \end{aligned}$$



Their “profits” **interact**



$$\begin{aligned}
 &\text{maximize}_{x^1} \quad \text{🍏} + 2 \text{🧀} - 2 \text{🍏} \text{🍏} - 3 \text{🧀} \text{🍫} \\
 &\text{s.t.} \quad 3 \text{🍏} + 4 \text{🧀} \leq 5, \\
 &\quad x^1 \in \{0, 1\}^2
 \end{aligned}$$

$$\begin{aligned}
 &\text{maximize}_{x^2} \quad 3 \text{🍏} + 5 \text{🍫} - 5 \text{🍏} \text{🍏} - 4 \text{🍫} \text{🧀} \\
 &\text{s.t.} \quad 2 \text{🍏} + 5 \text{🍫} \leq 5, \\
 &\quad x^2 \in \{0, 1\}^2
 \end{aligned}$$

Stable solutions



$$\begin{aligned} &\text{maximize}_{x^1 \in \{0,1\}^2} && x_1^1 + 2x_2^1 - 2x_1^1x_1^2 - 3x_2^1x_2^2 \\ &\text{s.t.} && 3x_1^1 + 4x_2^1 \leq 5, \end{aligned}$$

$$\begin{aligned} &\text{maximize}_{x^2 \in \{0,1\}^2} && 3x_1^2 + 5x_2^2 - 5x_1^2x_1^1 - 4x_2^2x_2^1 \\ &\text{s.t.} && 2x_1^1 + 5x_2^1 \leq 5, \end{aligned}$$



					
x_1^1	x_2^1	x_1^2	x_2^2		
0	0	0	0	0	0
1	0	0	0	1	0
0	1	0	0	2	0
0	0	1	0	0	3
0	0	0	1	0	5
1	0	1	0	-1	-2
0	1	0	1	-1	-1
0	1	1	0	2	3
1	0	0	1	1	5

Three **feasible strategies** for each player:
 $(x_1^i, x_2^i) \in \{(0, 0), (0, 1), (1, 0)\}$

But **only two** guarantee **stability**

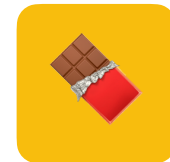


Stable solutions



$$\begin{aligned} &\underset{x^1 \in \{0,1\}^2}{\text{maximize}} && x_1^1 + 2x_2^1 - 2x_1^1x_1^2 - 3x_2^1x_2^2 \\ &\text{s.t.} && 3x_1^1 + 4x_2^1 \leq 5, \end{aligned}$$

$$\begin{aligned} &\underset{x^2 \in \{0,1\}^2}{\text{maximize}} && 3x_1^2 + 5x_2^2 - 5x_1^2x_1^1 - 4x_2^2x_2^1 \\ &\text{s.t.} && 2x_1^1 + 5x_2^1 \leq 5, \end{aligned}$$



x_1^1	x_2^1	x_1^2	x_2^2		
0	0	0	0	0	0
1	0	0	0	1	0
0	1	0	0	2	0
0	0	1	0	0	3
0	0	0	1	0	5
1	0	1	0	-1	-2
0	1	0	1	-1	-1
0	1	1	0	2	3
1	0	0	1	1	5

But **only two** guarantee **stability**



Players cannot profitably deviate:

- If **blue** plays 🧀, it would get **-1** instead of **2**
- If **red** plays 🍫, it would get **-1** instead of **3**

A background image of a wind farm with several wind turbines in a field. The entire image is covered with a semi-transparent purple overlay. The text "Energy Markets" is centered in white.

Energy Markets



SolarCorp Inc.

Simultaneous
Game



Hydro Inc.

“Cournot Game”



Canada taxes and regulates the production



SolarCorp Inc.

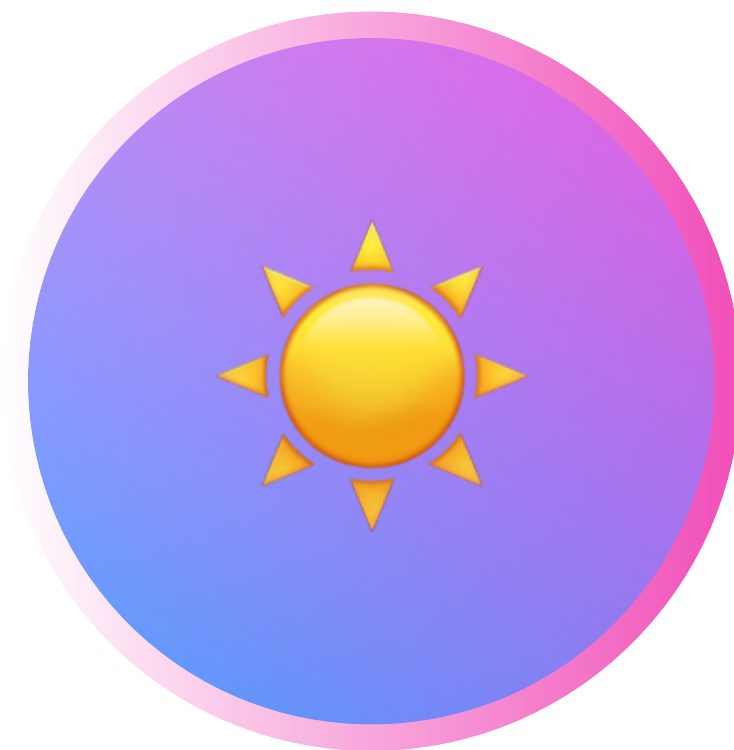
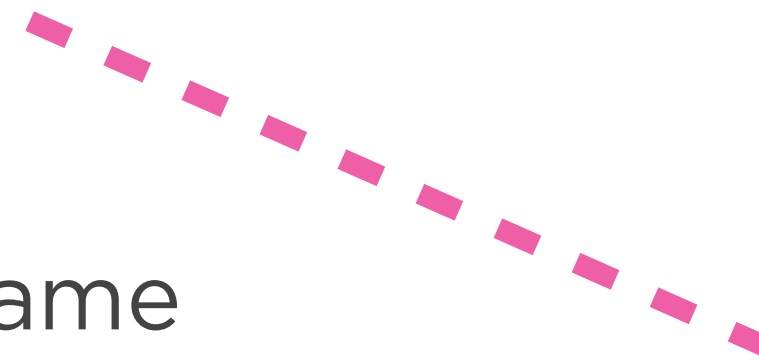
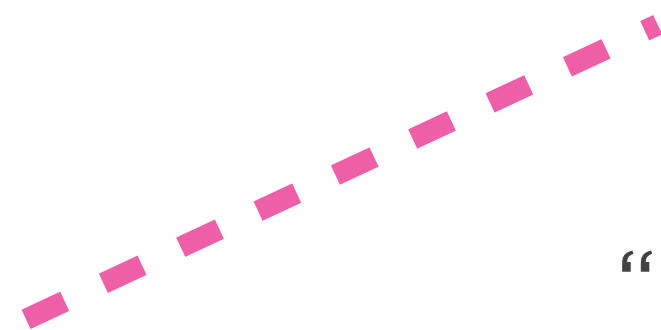
Simultaneous
Game



Hydro Inc.



Sequential
“Stackelberg” Game



SolarCorp Inc.



Simultaneous
Game



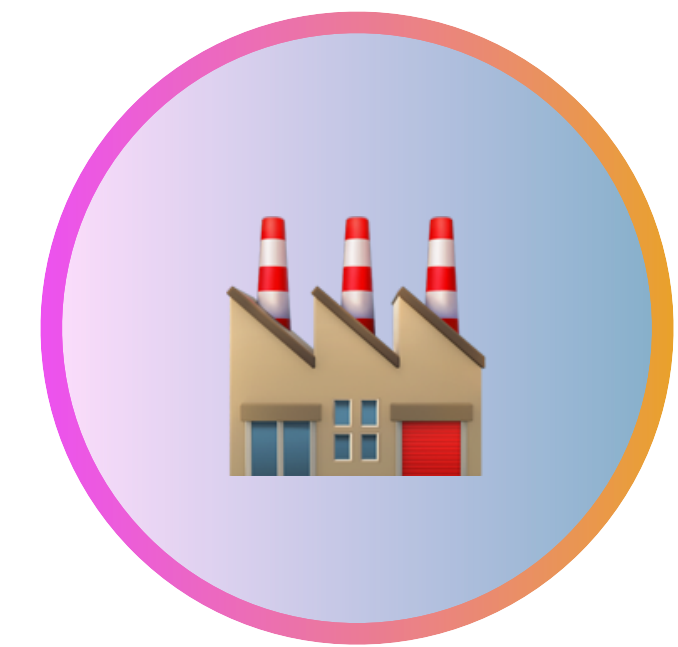
Hydro Inc.

Canada



Simultaneous
Game

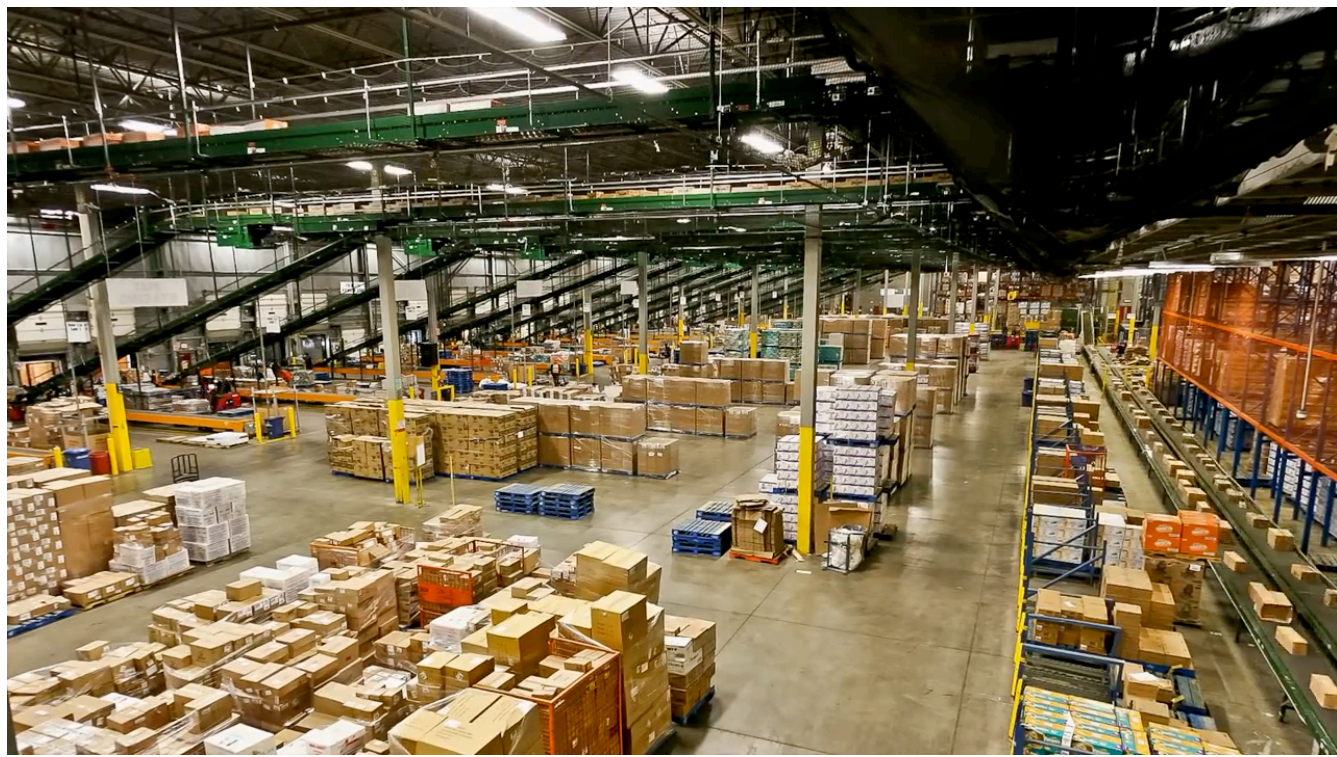
U.S.



This is a simultaneous game among **bilevel** (i.e., sequential) programs

And it can get more complex...

And it get more complex...



Supply Chain and Transportation

Cronert and Minner, 2021 (OR, TR-B)

Sagratella et al., 2020 (EJOR)

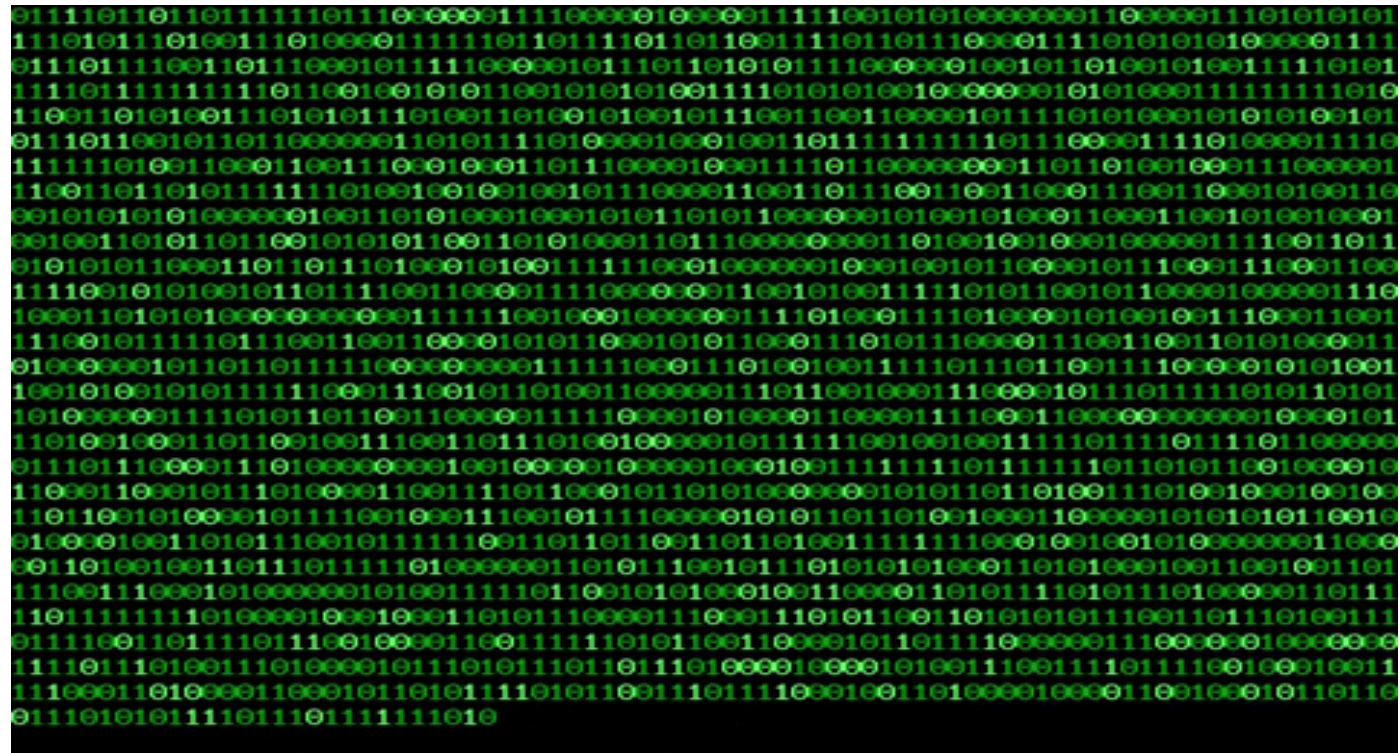
C. et al., 2018 (*IJ Production Economics*)



Simultaneous game among “bilevel” players

C. et al., 2023

(*Management Science*)



Cybersecurity

D. et al., 2023

(*Ericsson Inc, - Patent pending*)

Decision-making is rarely an individual task

It involves the **mutual interaction** of several **self-interested agents** and their **individual preferences**

Decision-making is rarely an individual task

What if agents (players) decide by solving **mixed-integer optimization problems**?

The Toolkit: Integer Programming Games

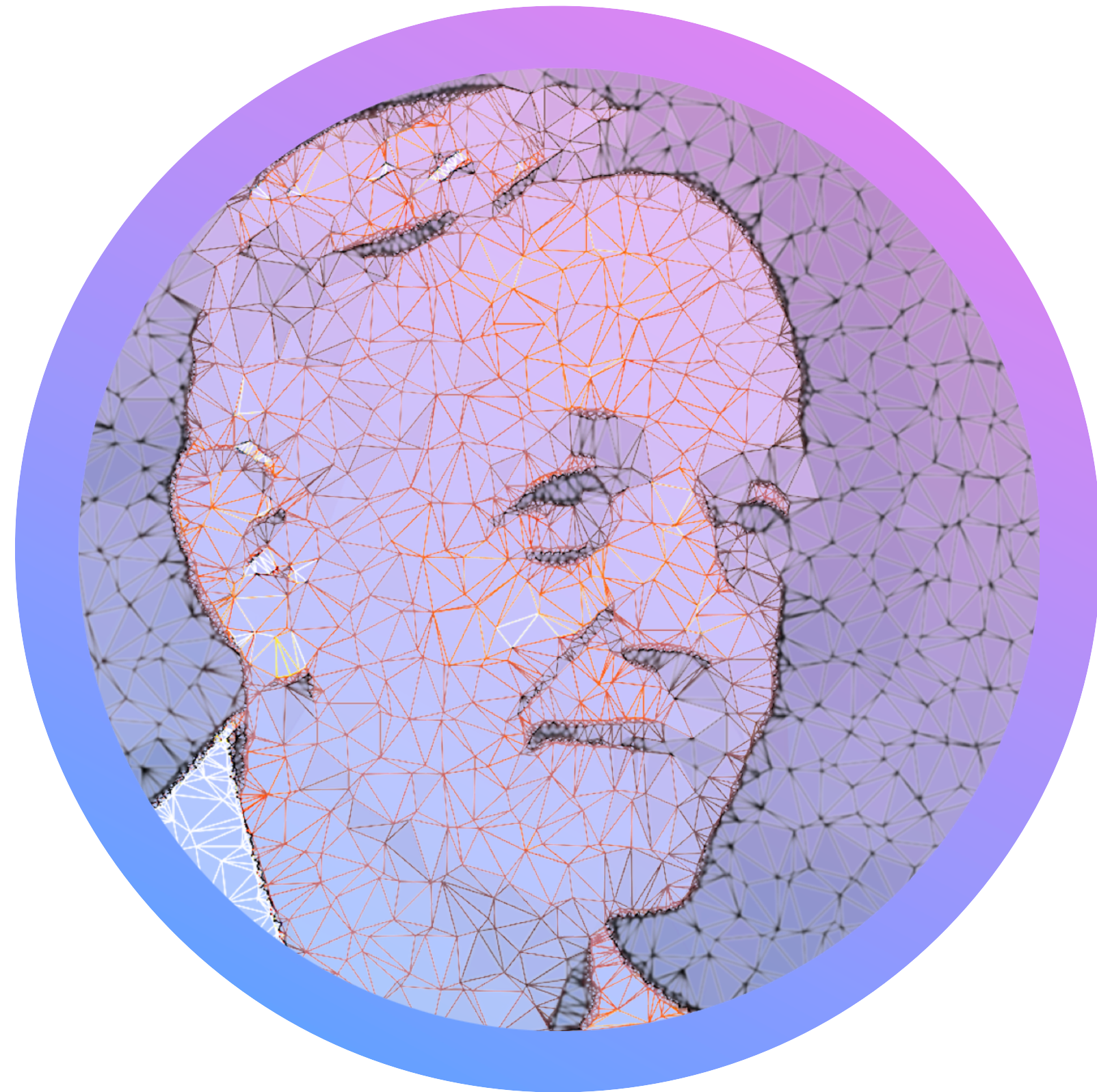
An Integer Programming Game (IPG) is a **simultaneous one-shot (static)** game among n players where each player $i = 1, \dots, n$ solves

$$\max_{x^i} \{ f^i(x^i; x^{-i}) : x^i \in \mathcal{X}^i \}$$

$$\mathcal{X}^i := \{ g^i(x^i) \leq b^i, \quad x^i \in \mathbb{Z}^{\alpha^i} \times \mathbb{R}^{\beta^i} \}$$

There is **common knowledge of rationality**, i.e., each player is **rational** and there is **complete information**

Stable solutions: Nash equilibria



$\bar{x} = (\bar{x}^1, \dots, \bar{x}^n)$ is a Pure Nash Equilibrium (**PNE**) if, for any player i ,

$$f^i(\bar{x}^i, \bar{x}^{-i}) \geq f^i(\hat{x}^i, \bar{x}^{-i}) \quad \forall \hat{x}^i \in \mathcal{X}^i$$

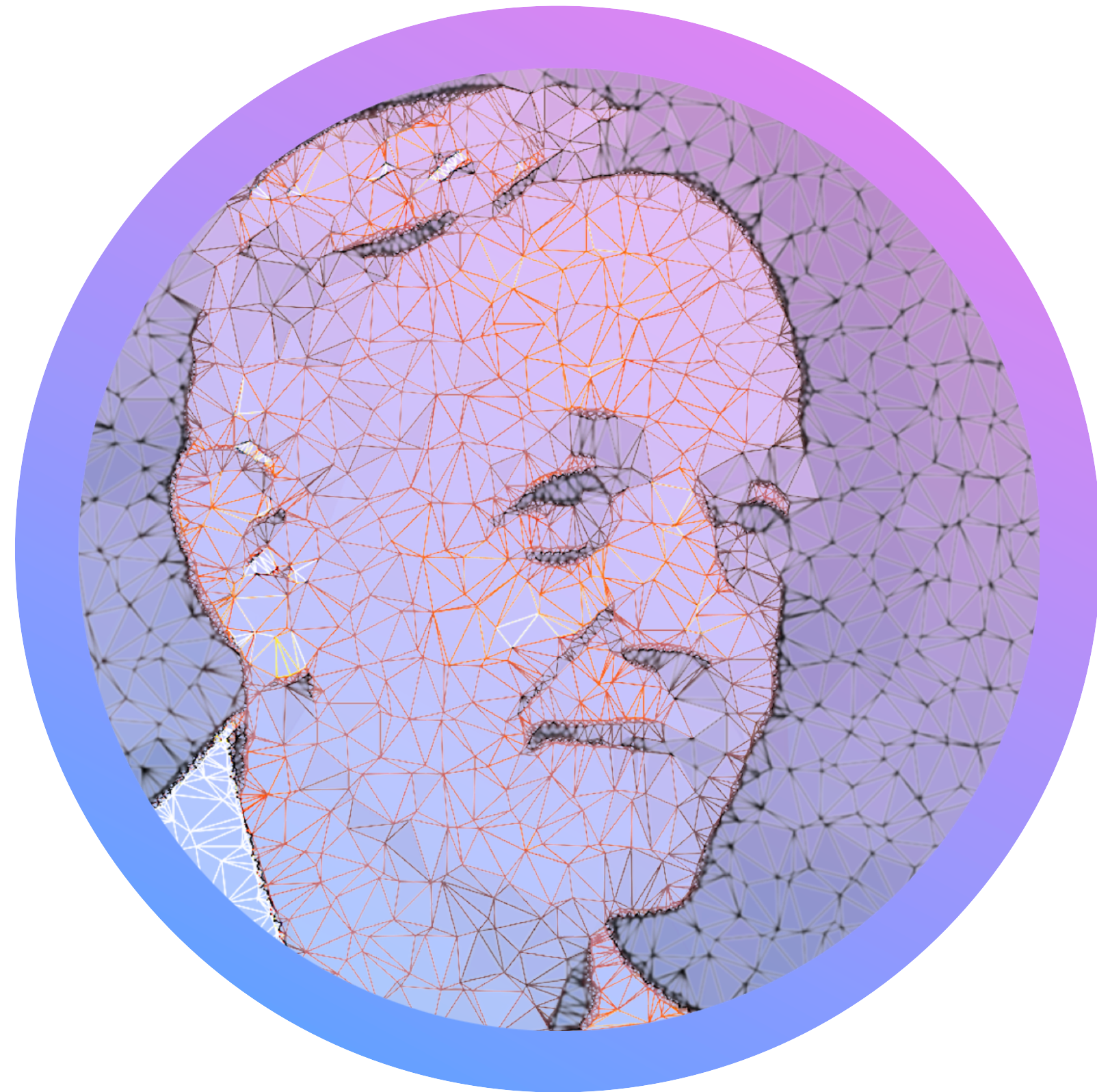
Mixed Nash
equilibrium

Players **randomize** over their pure strategies
i.e., probability distribution over \mathcal{X}^i

Approximate
equilibrium

Players can **deviate** up to an ϵ
e.g., relative or absolute deviation

Stable solutions: Nash equilibria



$\bar{x} = (\bar{x}^1, \dots, \bar{x}^n)$ is a *Pure Nash Equilibrium* (**PNE**) if, for any player i ,

$$f^i(\bar{x}^i, \bar{x}^{-i}) \geq f^i(\hat{x}^i, \bar{x}^{-i}) \quad \forall \hat{x}^i \in \mathcal{X}^i$$

PNEs and MNEs (C. et al.,

1. Deciding if an IPG has a pure equilibrium is Σ_2^p -complete
2. Deciding if an IPG has a mixed equilibrium is Σ_2^p -complete
3. If \mathcal{X}^i is finite for any player i , there exists a mixed equilibrium

Why?

Decision-making is rarely an individual task

What if agents (players) decide by solving **mixed-integer optimization problems**?

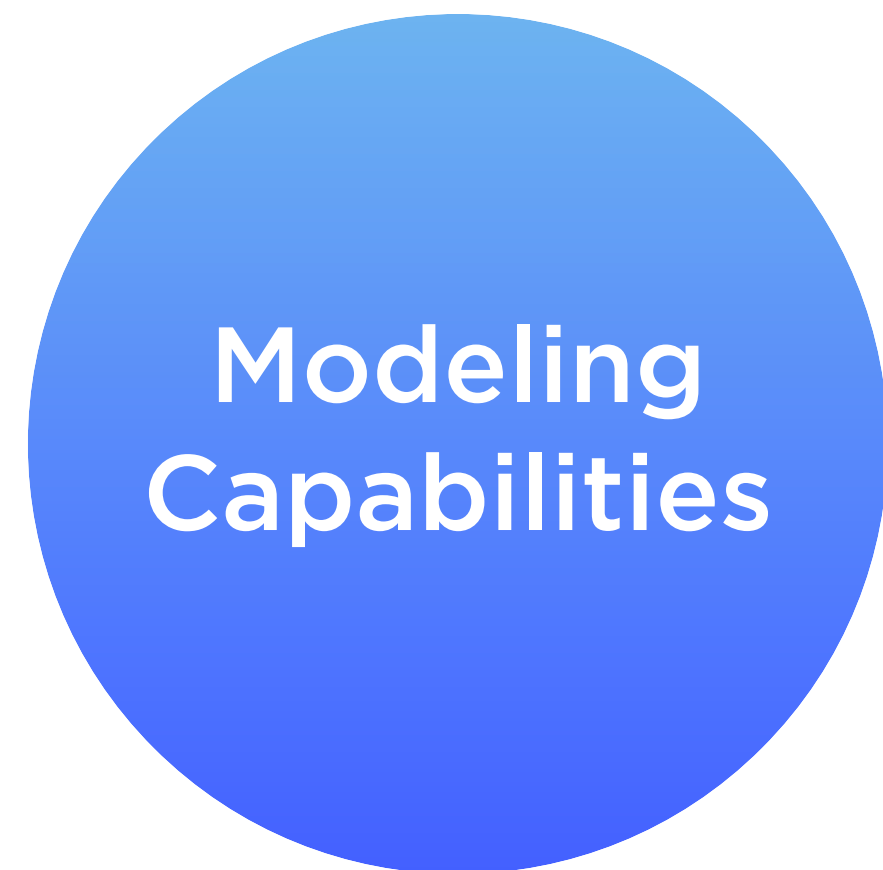


Modeling
Capabilities

Informative
Solutions

Practical
Impact

Why? Modeling Capabilities



They extend traditional **resource-allocation tasks and combinatorial optimization** problems to a multi-agent setting

Indivisible quantities, fixed-charge costs and logical implications often require discrete variables

Energy — Gabriel et al., 2013, David Fuller and Çelebi, 2017

Supply Chain — Anderson et al., 2017

Assortment-Price competitions — Federgruen and Hu, 2015

Kidney Exchange Problems — C. et al., 2017

Cybersecurity — D. et al., 2023

Why? Informative Contents of Equilibria

Econometrica, Vol. 70, No. 4 (July, 2002), 1341–1378

THE ECONOMIST AS ENGINEER: GAME THEORY, EXPERIMENTATION, AND COMPUTATION AS TOOLS FOR DESIGN ECONOMICS¹

BY ALVIN E. ROTH²

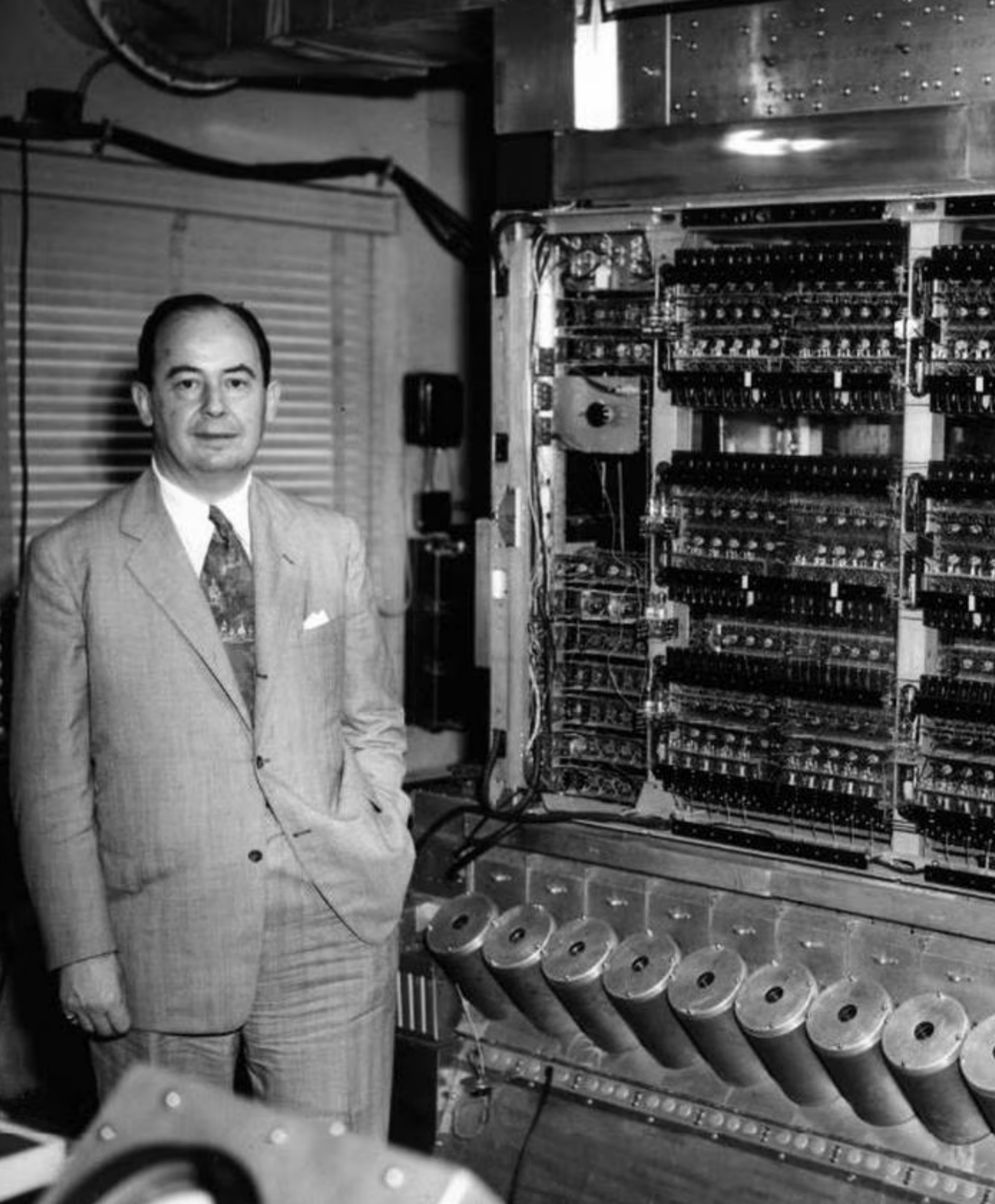


Informative
Solutions

*“Designers therefore **cannot work** only **with the simple conceptual models** used for theoretical insights into the general working of markets. Instead, **market design calls for an engineering approach**.*

***Experimental and computational economics** are natural complements to game theory in the work of design.”*

How?



The bad news: non-convexity

$$\mathcal{X}^i := \{g^i(x^i) \leq b^i, \quad x^i \in \mathbb{Z}^{\alpha^i} \times \mathbb{R}^{\beta^i}\}$$

Historically, **convexity** played a central role in shedding light on the **existence and computation** of Nash equilibria

State-of-the-art

		Payoff Types f^i	Constraints χ^i
Sagrattella, 2016	Branching Method	Convex payoffs	Bounded Convex Integer
C. et al., 2022	Sample Generation Method	Separable Payoffs	Bounded Mixed-Integer Linear
Schwarze and Stein, 2022	Branch-and-Prune	Quadratic Payoffs	Bounded Convex-Integer
C. et al., 2021	Cut-And-Play	Separable Payoffs	Polyhedral Convex-hull
Cronert and Minner, 2021	Exhaustive Sample Generation Method	Separable Payoffs	Bounded Pure-Integer
D. and Scatamacchia, 2023	Zero Regrets	Linearizable Payoffs	Bounded Mixed-Integer Linearizable

A 3-Phase Process

The **Approximate-Play-Improve** cycle



A 3-Phase Process

INPUT A game G described by the payoffs $f^i(x^i; x^{-i})$ and the actions \mathcal{X}^i

1

Approximate

Create \tilde{G} : Approximate \mathcal{X}^i with some set $\tilde{\mathcal{X}}^i$

2

Play

Compute a solution \bar{x} to \tilde{G} by solving an **optimization problem**

3

Improve

Check if \bar{x} is a Nash equilibrium

If not, refine at least a $\tilde{\mathcal{X}}^i$ and go to **2**
Otherwise: **return the Nash equilibrium \bar{x}**

Phase 1: Approximate

1

Approximate

Create \tilde{G} : Approximate \mathcal{X}^i with some set $\tilde{\mathcal{X}}^i$

The approximation $\tilde{\mathcal{X}}^i$ *should* exhibit **desirable properties** (e.g., convexity)

Computing the equilibria in \tilde{G} is **relatively easier** compared to G

However, the approximation often **changes the structure of equilibria**

Outer Approximation

$$\tilde{\mathcal{X}}^i \supseteq \mathcal{X}^i$$

Inner Approximation

$$\tilde{\mathcal{X}}^i \subseteq \mathcal{X}^i$$

Phase 2: Play

2

Play

Compute a solution \bar{x} to \tilde{G} by solving an **optimization problem**

Compute a tentative equilibrium \bar{x} for the approximated game \tilde{G}

Naturally depends on: the “***desirable properties***” of Phase 1

If $\tilde{\mathcal{X}}^i$ is convex, and f^i is concave in $x^i \rightarrow$ **Complementarity Problem**

Phase 3: Improve

3

Improve

Check if \bar{x} is a Nash equilibrium.

If not, refine at least a \tilde{x}^i and go to 2
Otherwise: **return the Nash equilibrium \bar{x}**

“*Separation theorem*” on steroids for Nash equilibria

Answer **two** questions:

Is \bar{x}^i a **feasible strategy with respect to x^i** ?

Is \bar{x} a **feasible Nash equilibrium**?

If one answer is negative, it provides a “**proof**”

Phase 3: Improve

3

Improve

Check if \bar{x} is a Nash equilibrium

If not, refine at least a \tilde{x}^i and go to 2
Otherwise: **return the Nash equilibrium \bar{x}**

Is \bar{x} a **feasible Nash equilibrium**?

Stability Step

No player should deviate from \bar{x}^i given \bar{x}^{-i}

$$\tilde{x}^i = \arg \max_{x^i} \{f^i(x^i, \bar{x}^{-i}) : x^i \in \mathcal{X}^i\}$$

No **iff** $f^i(\bar{x}^i; \bar{x}^{-i}) < f^i(\tilde{x}^i; \bar{x}^{-i})$

Is \bar{x}^i a **feasible strategy with respect to \mathcal{X}^i** ?

Membership Step

If $\bar{x}^i \notin \mathcal{X}^i$, then refine \tilde{x}^i

A 3-Phase Process

INPUT A game G described by the payoffs $f^i(x^i; x^{-i})$ and the actions \mathcal{X}^i

1

Approximate

Create \tilde{G} : Approximate \mathcal{X}^i with some set $\tilde{\mathcal{X}}^i$

2

Play

Compute a solution \bar{x} to \tilde{G} by solving an **optimization problem**

3

Improve

Check if \bar{x} is a Nash equilibrium

If not, refine at least a $\tilde{\mathcal{X}}^i$ and go to **2**
Otherwise: **return the Nash equilibrium \bar{x}**

An example: Cut-and-Play

INPUT A game G described by the **separable** payoffs $f^i(x^i; x^{-i})$ and the actions \mathcal{X}^i

1

Approximate

Create \tilde{G} : Approximate \mathcal{X}^i with some **polyhedron** $\tilde{\mathcal{X}}^i$

2

Play

Compute a solution \bar{x} to \tilde{G} via a **complementarity problem**

3

Improve

Check if \bar{x} is a Nash equilibrium

If not, refine at an $\tilde{\mathcal{X}}^i$ via **branching/cutting**

2

Otherwise: **return the Nash equilibrium \bar{x}**

The Challenges

And some challenges



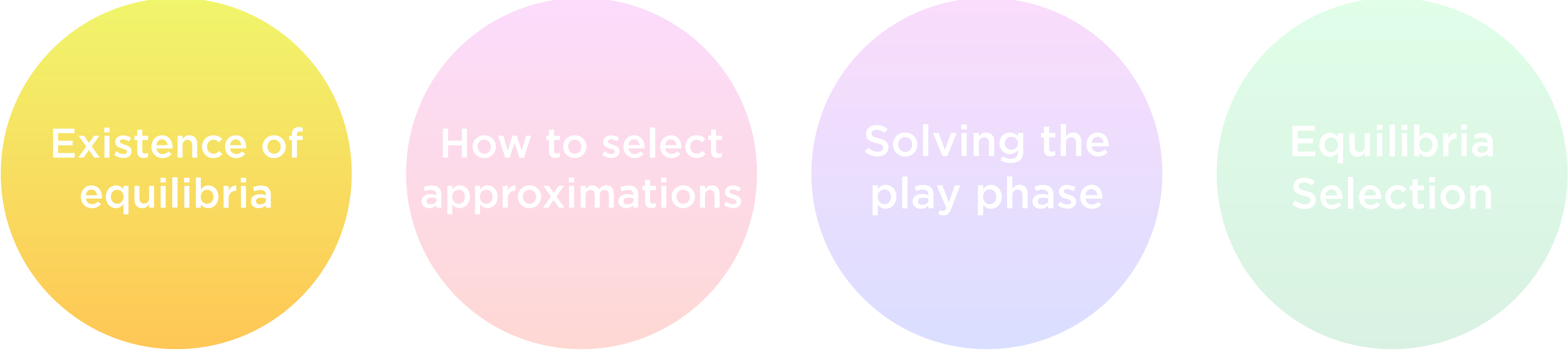
Existence of equilibria

How to select approximations

Solving the play phase

Equilibria Selection

And some challenges



Existence of
equilibria

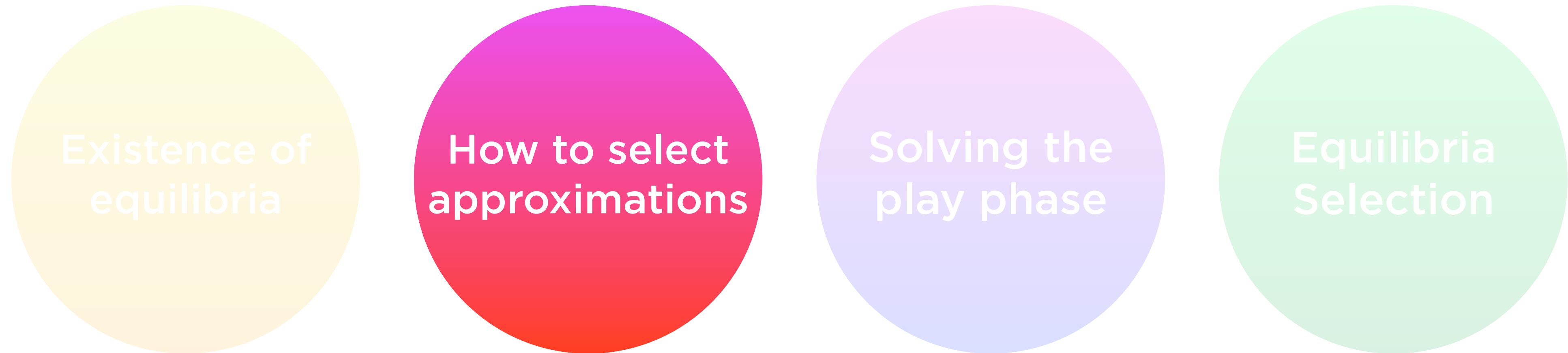
How to select
approximations

Solving the
play phase

Equilibria
Selection

An equilibrium **might not exist** in a given game G .
How to detect infeasibility?

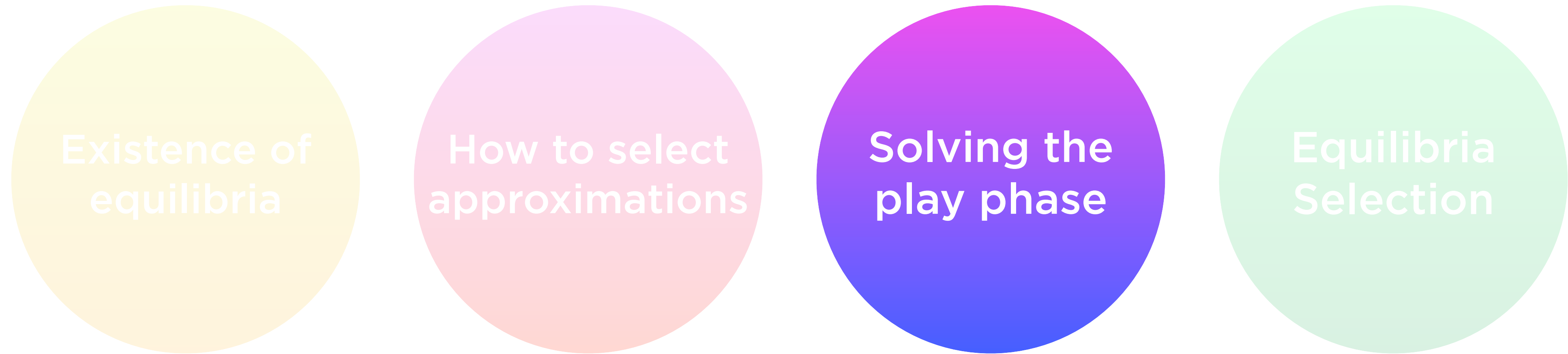
And some challenges



An equilibrium **might not exist** in an approximation \tilde{G} while one exists in the original game G (and vice versa)

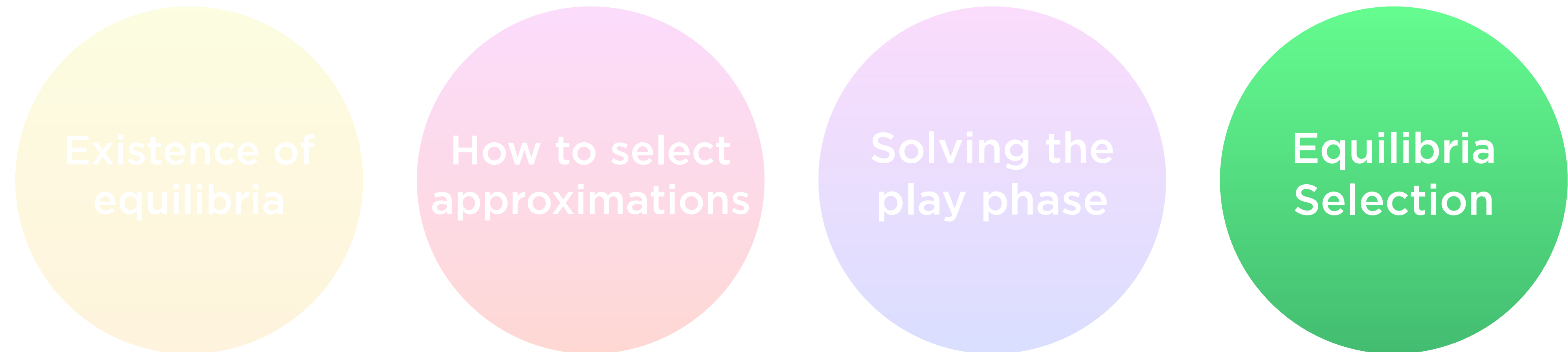
Outer vs Inner approximations

And some challenges



This phase often involves solving a **hard optimization problem**, e.g., a mixed-integer program, variational inequality or complementarity problem

And some challenges



If multiple equilibria exist, **selecting (i.e., optimizing) a specific one** is often complex from an algorithmic-design perspective

State-of-the-art

		Enumerate	Optimize	Payoff Types f^i	Constraints \mathcal{X}^i
Sagrattella, 2016	Outer Approximation	✓	✗	Convex payoffs	Bounded Convex Integer
C. et al., 2022	Inner Approximation	✗	✗	Separable Payoffs	Bounded Mixed-Integer Linear
Schwarze and Stein, 2022	Outer Approximation	✓	✗	Quadratic Payoffs	Bounded Convex-Integer
C. et al., 2021	Outer Approximation	✗	✗	Separable Payoffs	Polyhedral Convex-hull
Cronert and Minner, 2021	Inner Approximation	✓	✗	Separable Payoffs	Bounded Pure-Integer
D. and Scatamacchia, 2023	Outer Approximation	✓	✓	Linearizable Payoffs	Bounded Mixed-Integer Linearizable

CPU[|||||||||||||||||||||100.00%] Tasks: 139, 239 thr; 5 running
Mem[|||||||||||||||||617M/973M] Load average: 6.58 4.67 2.92
Swp[|||||||||432M/1022M] Uptime: 00:30:55

PID	USER	PRI	NI	VRT	RES	SHR	S	CPU%	MEM%	TIME+	Command
4828	netcon1	20	0	655M	24200	10940	R	13.0	2.4	1:12.26	/usr/lib/gnome-t
32109	netcon1	20	0	78392	45132	2688	S	11.0	4.5	0:23.35	tmux -u -2 -f /u
898	root	20	0	395M	44868	14160	S	3.9	4.5	0:57.78	/usr/lib/xorg/Xo
4523	netcon1	20	0	1153M	36200	13844	S	1.9	3.6	0:19.72	complz
41994	netcon1	39	19	10832	932	816	S	0.6	0.1	0:00.07	cmatrix -b
32216	netcon1	20	0	34032	4720	2976	R	0.6	0.5	0:01.18	htop
4207	netcon1	20	0	356M	2884	1772	S	0.0	0.3	0:02.29	/usr/bin/ibus-da
4512	netcon1	20	0	496M	4744	3968	S	0.0	0.5	0:01.36	/usr/bin/vmtools
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32354	netcon1	39	19	40940	12196	4728	S	0.0	1.2	0:00.77	/usr/bin/python
4295	netcon1	20	0	203M	364	28	S	0.0	0.0	0:00.74	/usr/lib/ibus/ib
4269	netcon1	20	0	356M	2884	1772	S	0.0	0.3	0:01.58	/usr/bin/ibus-da
32406	netcon1	39	19	33016	3368	3000	S	0.0	0.3	0:00.29	bmon --show-all
4297	netcon1	20	0	203M	364	28	S	0.0	0.0	0:00.60	/usr/lib/ibus/ib
32477	netcon1	39	19	4776	1928	1512	S	0.0	0.2	0:00.06	/bin/sh /usr/bin
32338	netcon1	39	19	4508	1580	1476	S	0.0	0.2	0:00.05	/bin/sh /usr/bin
4272	netcon1	20	0	468M	4360	1972	S	0.0	0.4	0:00.95	/usr/lib/ibus/ib
32286	netcon1	39	19	19720	2964	2628	S	0.0	0.3	0:00.15	/bin/bash /usr/b
4426	netcon1	9	-13	431M	6396	4784	S	0.0	0.6	0:09.21	/usr/bin/pulseau
4281	netcon1	20	0	468M	4360	1972	S	0.0	0.4	0:00.62	/usr/lib/ibus/ib
32198	netcon1	20	0	1964M	2892	2628	S	0.0	0.3	0:00.08	/bin/bash /usr/b
4439	netcon1	-6	0	431M	6396	4784	S	0.0	0.6	0:09.21	/usr/bin/pulseau
32313	netcon1	39	19	4508	1580	1476	S	0.0	0.2	0:00.05	/bin/sh /usr/bin
3381	root	20	0	188M	2064	1736	S	0.0	0.2	0:01.14	/usr/bin/vmtools
720	root	20	0	291M	2848	2092	S	0.0	0.3	0:00.46	/usr/lib/account
4369	netcon1	20	0	368M	2308	1340	S	0.0	0.2	0:00.13	/usr/lib/x86_64-
4386	netcon1	20	0	666M	1992	864	S	0.0	0.2	0:00.06	/usr/lib/x86_64-
891	root	20	0	357M	1688	1216	S	0.0	0.2	0:00.03	/usr/sbin/lightd
4691	netcon1	20	0	349M	868	0	S	0.0	0.1	0:00.16	/usr/bin/zeitgel
4334	netcon1	20	0	540M	2388	1488	S	0.0	0.2	0:00.39	/usr/lib/gnome-s
4828	netcon1	20	0	655M	24200	10940	S	0.0	2.4	0:00.34	/usr/lib/gnome-t
4705	netcon1	20	0	314M	2188	828	S	0.0	0.2	0:00.11	/usr/lib/x86_64-
4699	netcon1	20	0	314M	2188	828	S	0.0	0.2	0:00.17	/usr/lib/x86_64-
4139	netcon1	20	0	43780	1852	728	S	0.0	0.2	0:00.94	dbus-daemon --fo
4754	netcon1	20	0	638M	1136	4	S	0.0	0.1	0:00.16	/usr/lib/x86_64-
4037	netcon1	20	0	207M	132	0	S	0.0	0.0	0:00.16	/usr/bin/gnome-k
4527	netcon1	20	0	1153M	36200	13844	S	0.0	3.6	0:00.30	complz
4352	netcon1	20	0	540M	2388	1488	S	0.0	0.2	0:00.28	/usr/lib/gnome-s
4205	netcon1	20	0	48356	140	0	S	0.0	0.0	0:00.04	upstart-file-bri
32361	netcon1	39	19	4508	1528	1424	S	0.0	0.2	0:00.01	/bin/sh /usr/bin
4693	netcon1	20	0	349M	868	0	S	0.0	0.1	0:00.11	/usr/bin/zeitgel
10780	netcon1	20	0	2387M	64696	12392	S	0.0	6.5	0:00.01	unity-control-ce
32523	netcon1	39	19	4508	1620	1468	S	0.0	0.2	0:00.01	/bin/sh /usr/bin
32416	netcon1	39	19	19704	2876	2568	S	0.0	0.3	0:00.03	/bin/bash /usr/b
747	root	20	0	291M	2848	2092	S	0.0	0.3	0:00.15	/usr/lib/account
4182	netcon1	20	0	207M	132	0	S	0.0	0.0	0:00.12	/usr/bin/gnome-k
4379	netcon1	20	0	770M	3536	2664	S	0.0	0.4	0:00.07	/usr/lib/x86_64-
4201	netcon1	20	0	39864	144	8	S	0.0	0.0	0:00.10	upstart-dbus-bri
4756	netcon1	20	0	638M	1136	4	S	0.0	0.1	0:00.08	/usr/lib/x86_64-

Protecting Critical Infrastructure

..io
..ll.

.....[394250].....

buntu1), gtk2-engines-nurrine:amd64 (0.98.2-0ubuntu2.2), xcursor-themes:amd64 (1.0.4-1), ibus-gtk:amd64 (1.5.11-1ubuntu2), libpan-gnome-keyringamd64 (3.18.3-0ubuntu2), libexempi3:amd64 (2.2.2-2), libgettext-o-dev:amd64 (0.19.7-2ubuntu3), ally-profile-manager-indicator:amd64

ECONNREFUSED 111 Connection refused
EEXIST 17 File exists
ENODATA 61 No data available

32KiBSpeedometer 2.8

TX: ens330 B/s0 B/s0 B/s

6 | \ - | x < W T 8 P 9 ' (_ : | Z U y V c Z) G 9 X H u +
/ i } W | Y 8 j d c l g t & # x U ; ^ H / " X - # [= L O O #
> H ^ } # | e k 3 (H | n x # * 4 a t ,] m p q H x 4 >
\$ F Z 4 Z J + n r : u F g D r V ; R & Q Z , > d o { S
/] & \$ P y H F s s > b V J 2 i a ^ p 6] l E = n a % &
w q a 2 T # ? L) } K D c C O (Q x % } t t / f B
c s e t u n s t l e a t 4 o s n

-precision valuemaximum number of significant digits to print
-quietsuppress all warning messages
-regard-warningspay attention to warning messages

Change: 2017-03-16 11:42:39.209110393 -0700
Birth: -

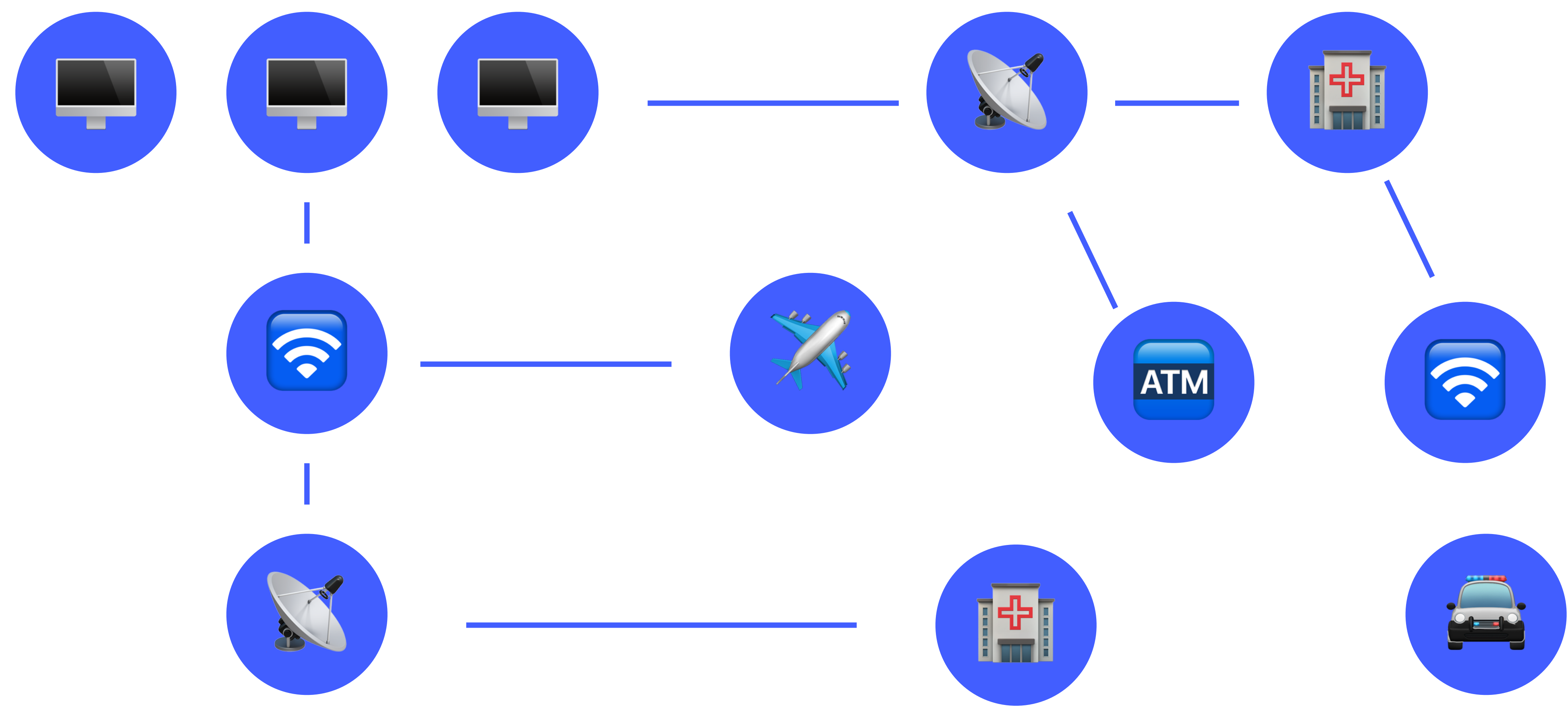
0 directories, 1 file

enmuacan (en-mu-ac-an) echo-november-mike-uniform-alfa-charlie-alfa-november
yagFetyot (yag-Fet-yot) yankee-alfa-golf-Foxtrot-echo-tango-yankee-oscar-tango

00000b7000 00 00 00 00 00 00 00 0030 03 00 00 12 00 00 00.....0.....
00000b8000 00 00 00 00 00 00 00 0000 00 00 00 00 00 00 00.....
00000b9073 02 00 00 12 00 00 00 0000 00 00 00 00 00 00 005.....

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Critical Node Game



Critical Node Game



An attacker gain access to the network
at **an unknown node** (e.g., server)

Critical Node Game



An attacker gain access to the network
at **an unknown node** (e.g., server)



The network operator decides **how to protect their network**

Critical Node Game



Critical Node Game

Players act simultaneously

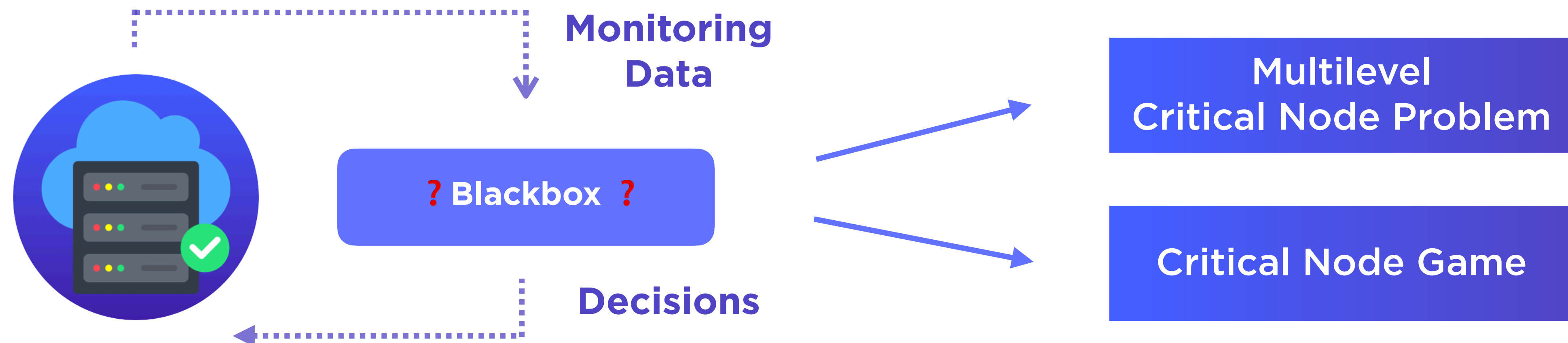
Multilevel
Critical Node Problem

Players act sequentially

Real-time deployment of protective resources

A-priori **security posture assessment**

Critical Node Game



Mathematical Models



Attacker

The attacker has a **budget** A

Attacking $i \in R$ **costs** c_i^a and gives a **benefit** p_i^a

The variable α_i is **1** if the attacker attacks $i \in R$



Defender

The defender has a **budget** D

Defending $i \in R$ **costs** c_i^d and gives a **benefit** p_i^d

The variable x_i is **1** if the defender protects $i \in R$

Mathematical Models



x_i

α_i

Payoff^D

Payoff^A

0

0

p_i^d

$-\gamma p_i^a$

$\gamma \in [0,1]$: **opportunity cost** of not attacking

0

1

δp_i^d

p_i^a

$\delta \in [0,1]$: degradation with **ongoing attack**

1

0

ϵp_i^d

0

$\epsilon \in [0,1]$: degradation with **passive mitigation**

1

1

ηp_i^d

$(1 - \eta)p_i^a$

$\eta \in [0,1]$: degradation with **active mitigation**

Mathematical Models

Critical Node Game

Simultaneous IPG

$$\max_x \left\{ f^d(x; \alpha) : d^\top x \leq D, x \in \{0, 1\}^{|R|} \right\}$$

$$\max_\alpha \left\{ f^a(\alpha; x) : a^\top \alpha \leq A, \alpha \in \{0, 1\}^{|R|} \right\}$$

Multilevel
Critical Node Problem

Bilevel Game

$$\begin{aligned} \max_{x, \hat{\alpha}} \quad & f^d(x, \hat{\alpha}) \\ \text{s.t.} \quad & d^\top x \leq D, \\ & x \in \{0, 1\}^{|R|}, \\ & \hat{\alpha} \in \arg \max_\alpha \left\{ f^a(\alpha; x) : a^\top \alpha \leq A, \alpha \in \{0, 1\}^{|R|} \right\}. \end{aligned}$$

Metrics

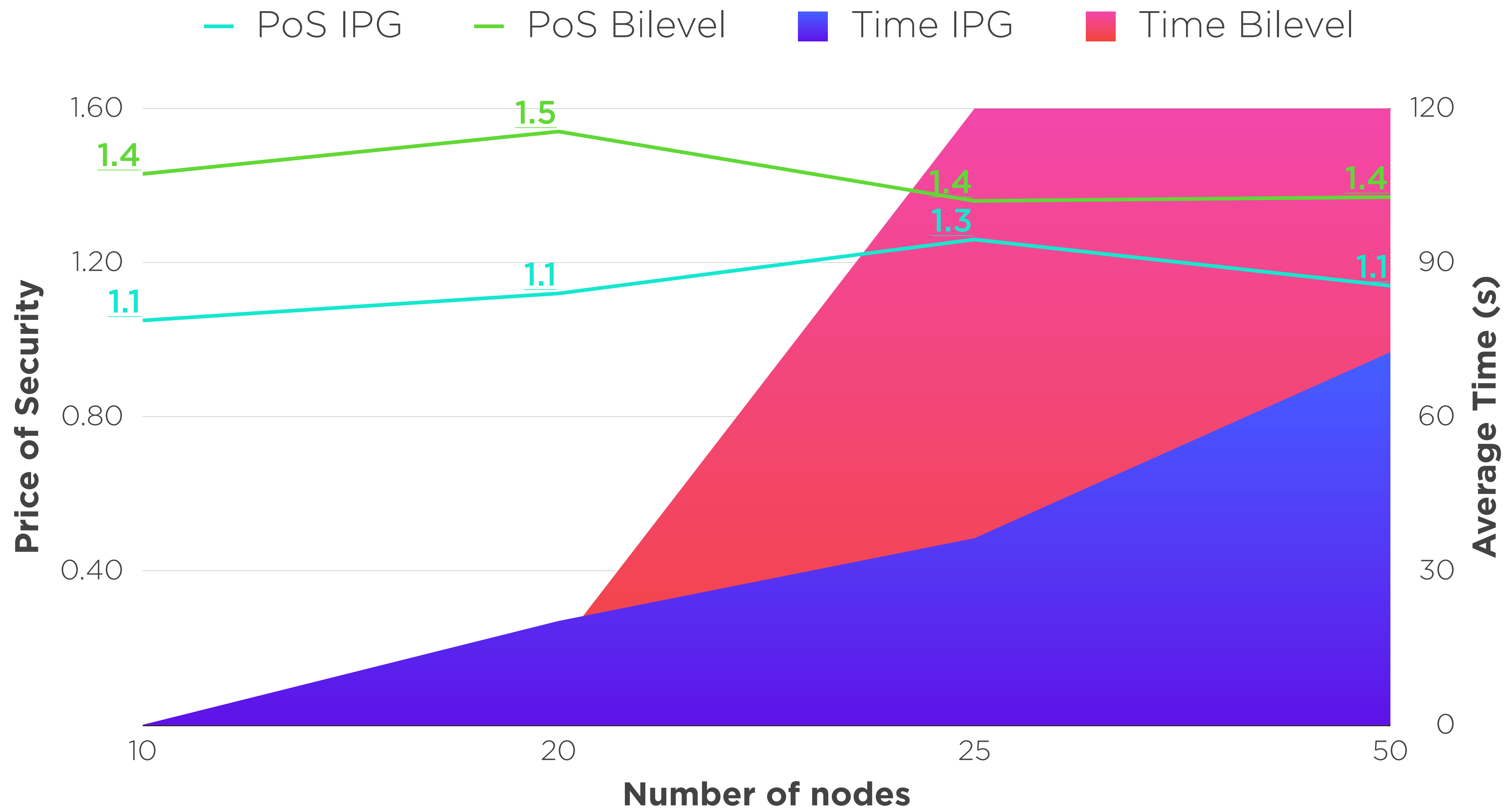
We measure time, and the **effectiveness of the defender's strategy**

$$\text{Price of Security}_{(\bar{x}, \bar{\alpha})} = \frac{\text{Best defender's objective for any outcome}}{\text{Defender's objective in } (\bar{x}, \bar{\alpha})}$$

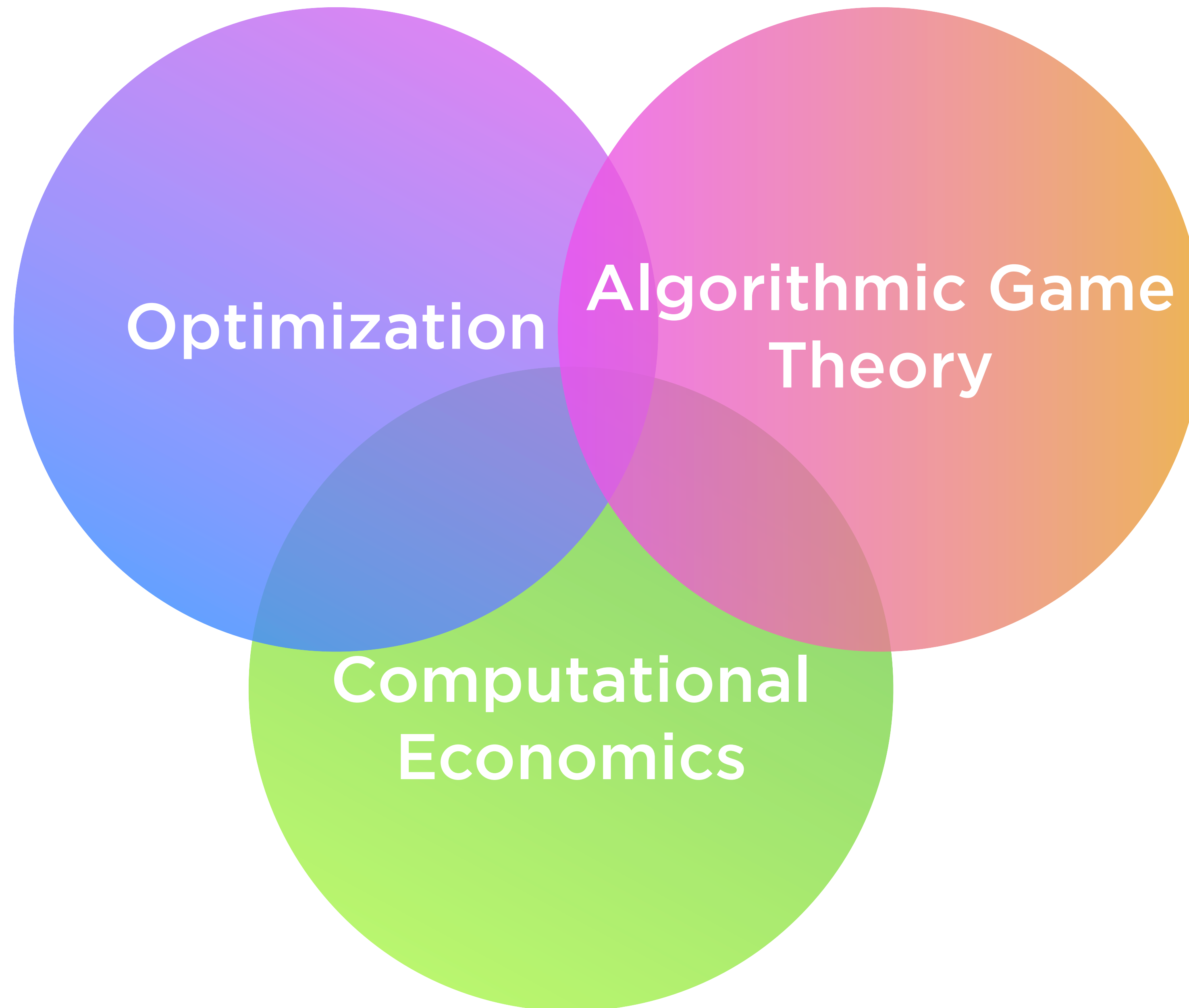
All the tests and algorithms of this tutorial are implemented in our open-source package **ZERO**

<https://github.com/ds4dm/ZERO>

Results




Looking Ahead



Decision-making is rarely an individual task

Toolbox to model competitive decision-making with **mixed-integer optimization**



Modeling
Capabilities

Informative
Solutions

Practical
Impact

There are still (too) many open questions

Some perspectives



Applications

Deployment of IPGs in **new application domains**

Develop **theoretically-grounded and efficient**
both general and problem-specific algorithms



Optimization



Fairness

Solutions **balancing** the decision-makers
selfishness with **societal goals**



Integer Programming Games: A Gentle Computational Overview

INFORMS TutORial in Operations Research, 2023



Margarida



Gabriele



Andrea



Sriram

arXiv 2303.11188