Integer Programming Games

Do You Really Need Them?

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13th Day on Computational Game Theory
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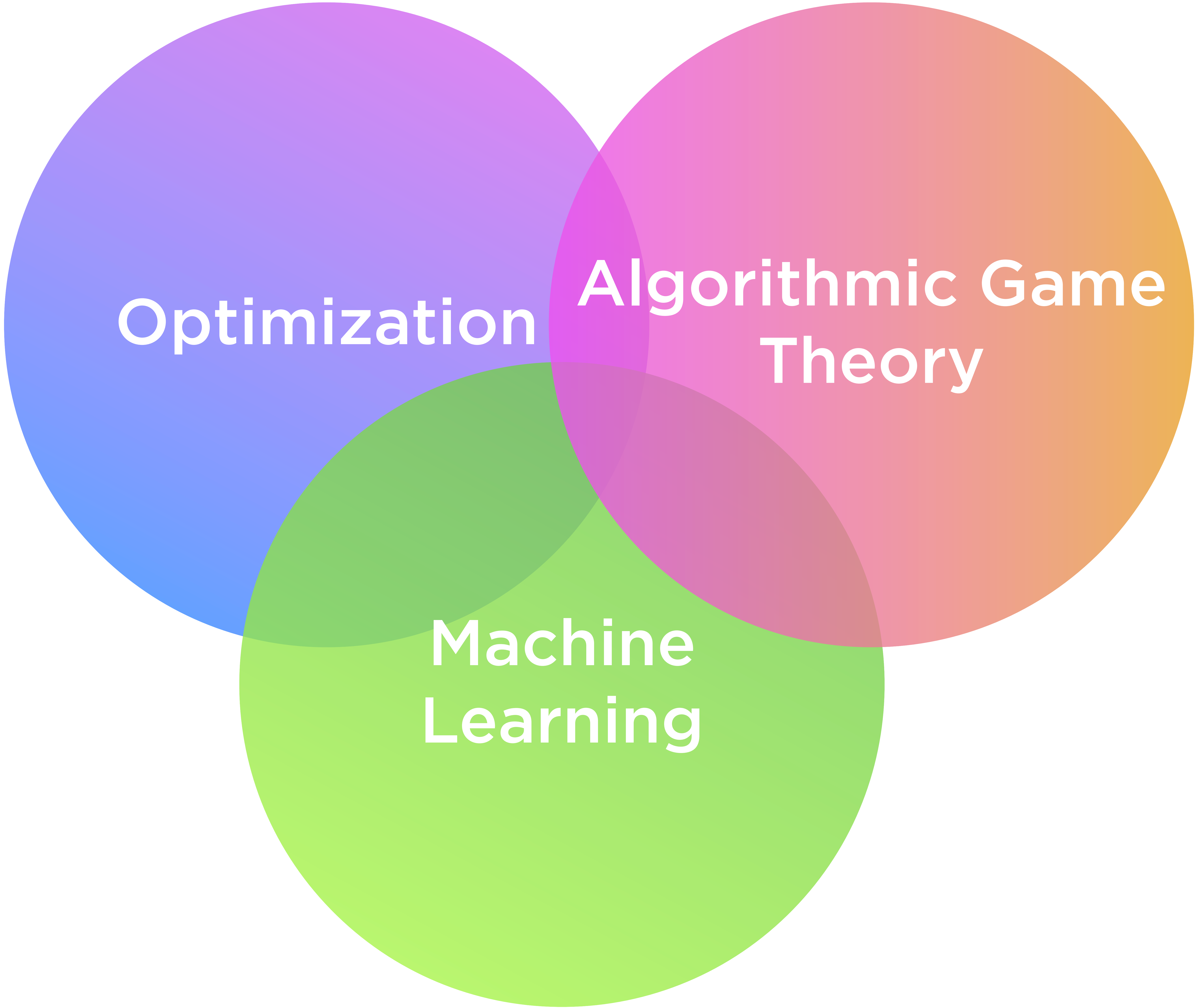
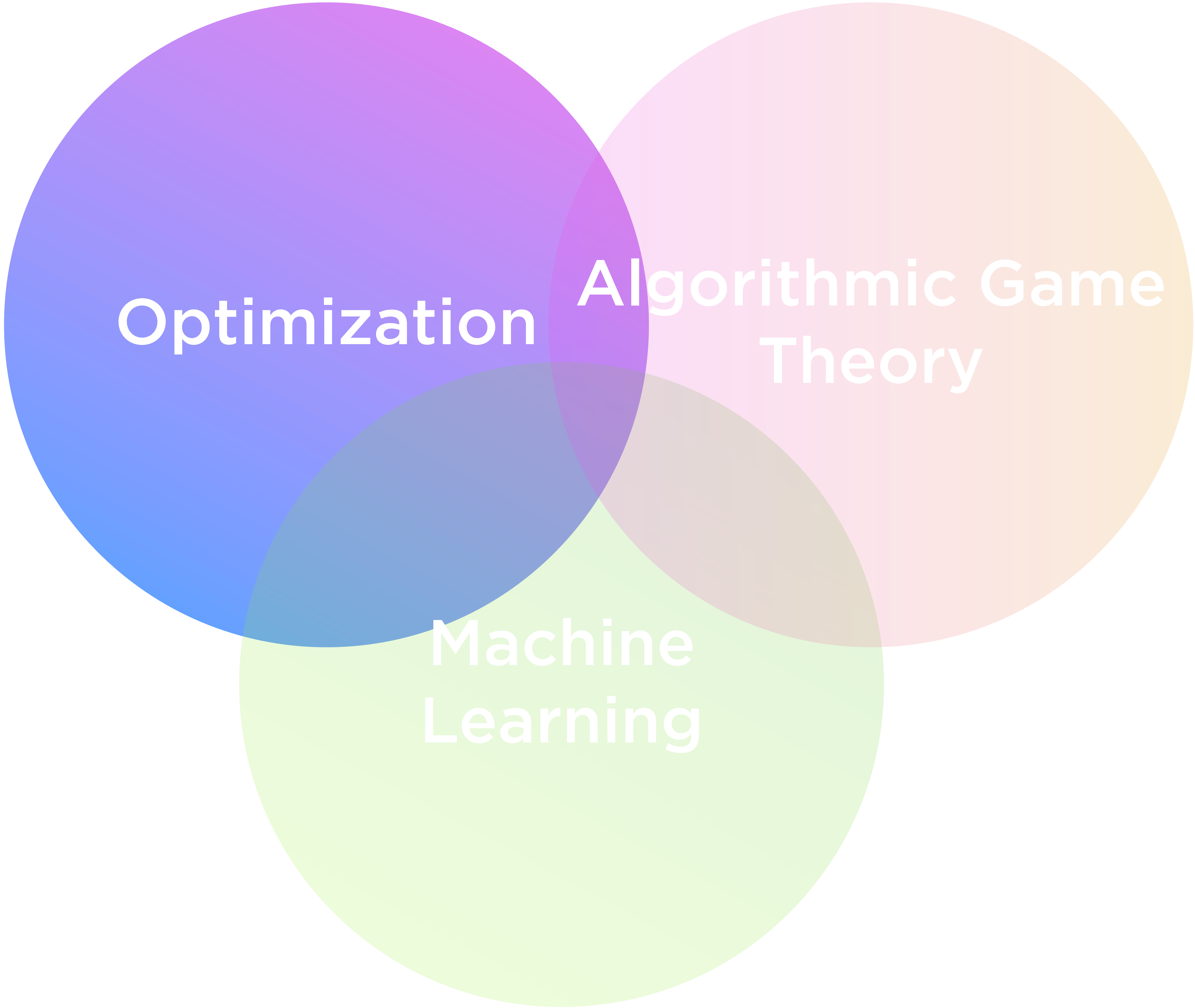

Optimization
Algorithmic Game Theory
Machine Learning


Optimization
Algorithmic Game Theory
Machine Learning

Rosario Scatamacchia
Politecnico di Torino

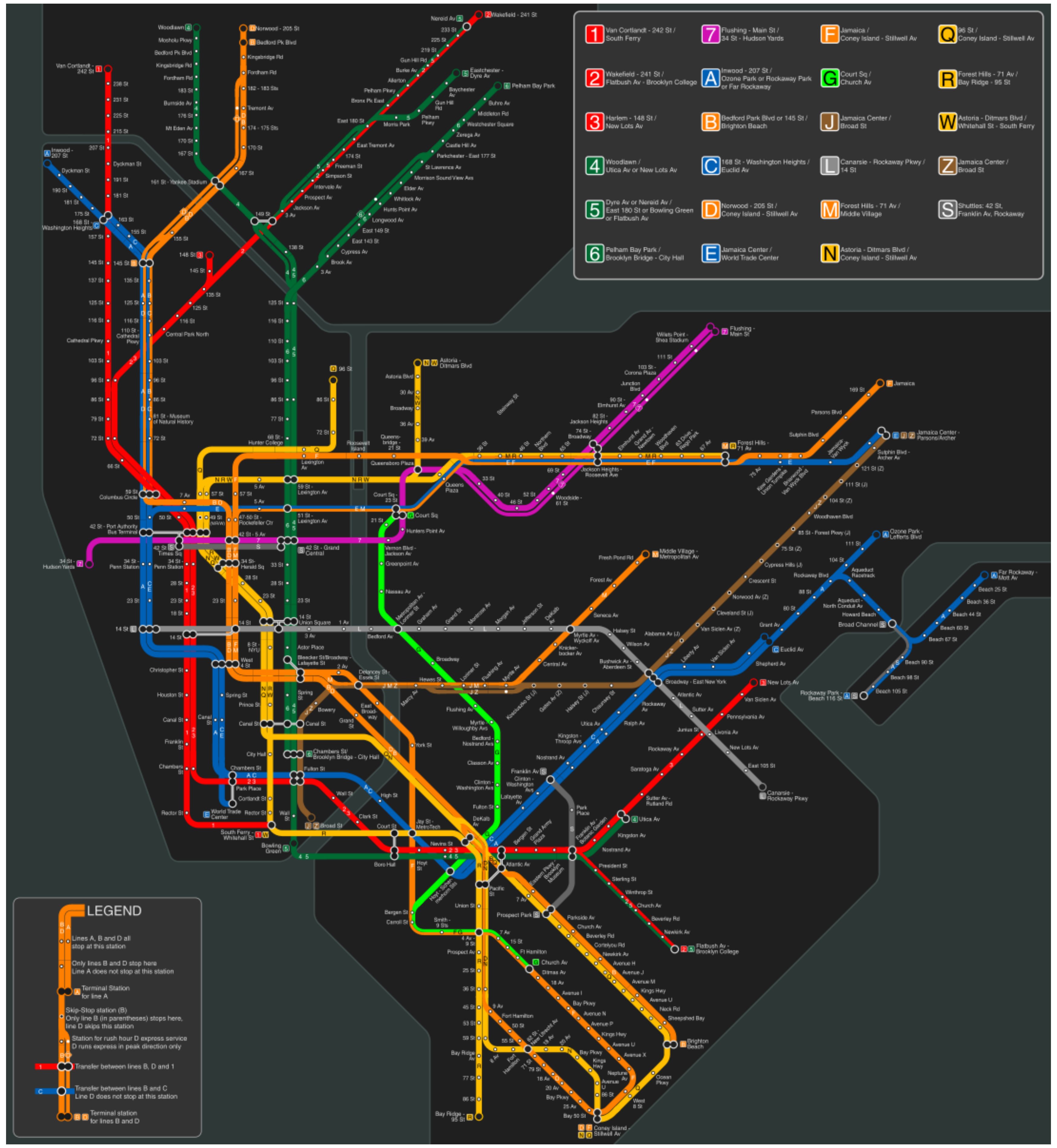
Andrea Lodi
Cornell and Cornell Tech

Margarida Carvalho
Universitè de Montréal

Sriram Sankaranarayanan
IIM Ahmedabad

Commuting

Network Formation



There are n players optimizing simultaneously the **shortest path** on a network, and want to **share** the **setup costs**

Choices of **other players**

$$\min_{x^i} \{ u^i(x^i; x^{-i}) : x^i \in \mathcal{X}^i \}$$

Choices of **player** i

How do we **algorithmically compute** the **best stable outcome?**

Network Congestion

A regulator wants to **intervene in the game**

Nash equilibria as proxy of rational behavior
“Economists as Engineers” *(Roth, 2002)*

Multi-agent Assortment

$$
\begin{aligned}
\max_{x^1} \quad & 6x_1^1 + x_2^1 \\
\text{s.t.} \quad & 3x_1^1 + 2x_2^1 \leq 4 \\
& x^1 \in \{0,1\}^2
\end{aligned}
$$

Their "profits" **interact**

$$\begin{aligned} \max_{x^1} \quad & 6x_1^1 + x_2^1 - 4x_1^1 x_1^2 + 6x_2^1 x_2^1 \\ \text{s.t.} \quad & 3x_1^1 + 2x_2^1 \leq 4 \\ & x^1 \in \{0,1\}^2 \end{aligned}$$

$$\begin{aligned} \max_{x^2} \quad & 4x_1^2 + 2x_2^2 - x_1^2 x_1^1 - x_2^2 x_2^1 \\ \text{s.t.} \quad & 2x_1^2 + 3x_2^2 \leq 4 \\ & x^2 \in \{0,1\}^2 \end{aligned}$$

Knapsack Games (Carvalho et al., 2022; ***D.*** *and Scatamacchia, 2022), Critical Node Game (**D.** et al., 2023)*

And it can get more complex…

And it get more complex...

Facility Location and Design Game

Cronert and Minner, 2021

(Operations Research)

Simultaneous game among "bilevel" players

*Carvalho, **D.** et al, 2023*

(Management Science)

Cybersecurity

***D.** et al, 2023*

(Ericsson Inc, - Patent pending)

Decision-making is rarely an individual task

Self-driven interactions with other decision-makers

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Self-driven interactions with other decision-makers

deciding by solving **complex optimization problems**

- Flexible Modeling
- Equilibria Computation
- Practical Insights from Equilibria

The Toolkit: Integer Programming Games

An Integer Programming Game (IPG) is a *simultaneous one-shot (static)* game among n players where each player $i = 1, \ldots, n$ solves

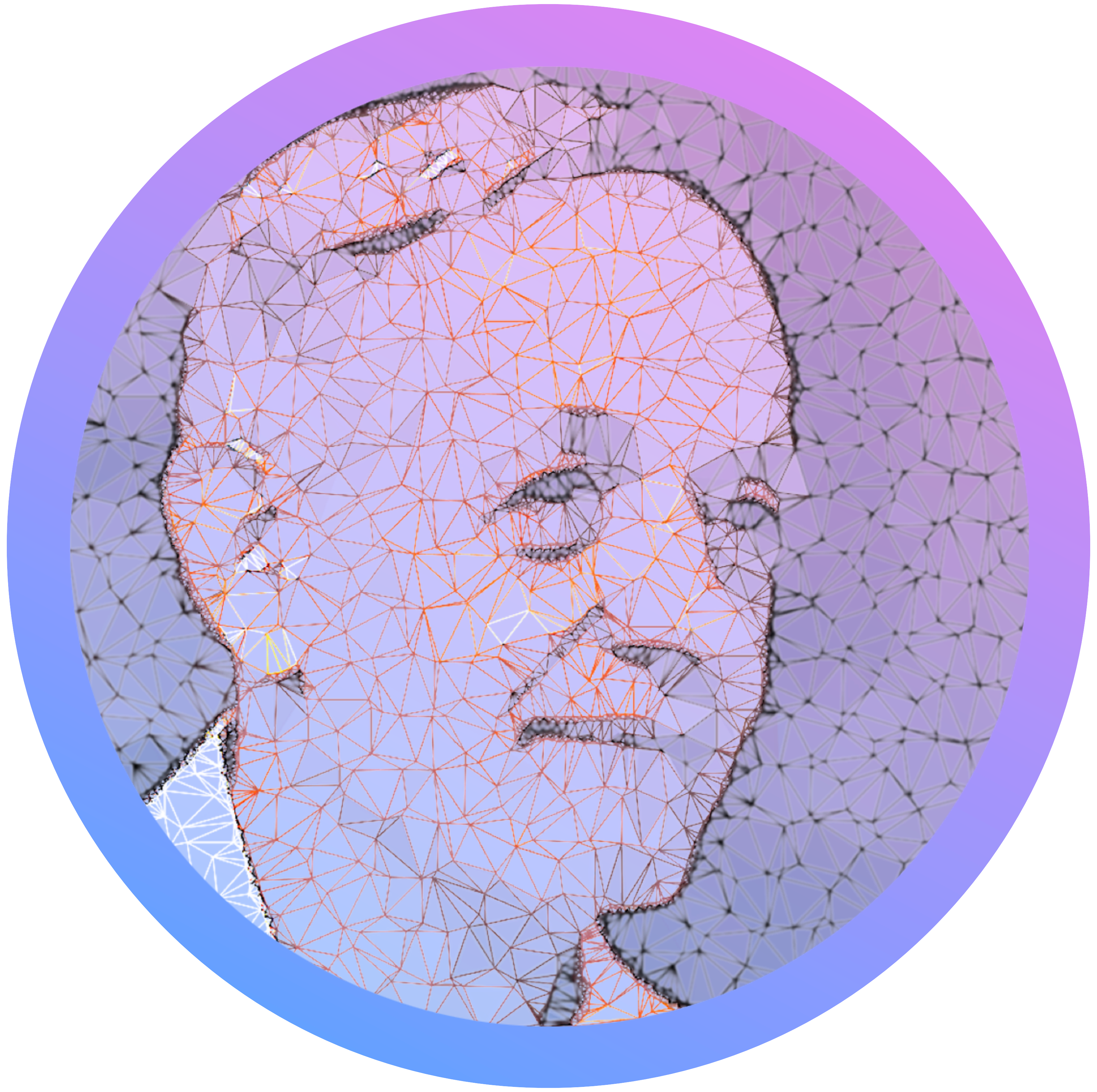
$$\min_{x^i} \{u^i(x^i; x^{-i}) : x^i \in \mathcal{X}^i\}$$

$$\mathcal{X}^i := \{A^i x^i \leq b^i, \;\; x^i \in \mathbb{Z}^{\alpha^i} \times \mathbb{R}^{\beta^i}\}$$

There is **common knowledge of rationality**, i.e., each player is **rational** and there is **complete information**

Köppe et al., (2011), Sagratella (2015), ***D.*** *et al (2021),* **D.** *(2022)*

Nash equilibria



$\bar{x} = (\bar{x}^1, \ldots, \bar{x}^n)$ *is a Pure Nash Equilibrium (***PNE***) if, for any player* i*,*

$$u^i(\bar{x}^i, \bar{x}^{-i}) \leq u^i(\hat{x}^i, \bar{x}^{-i}) \quad \forall \hat{x}^i \in \mathcal{X}^i$$

PNEs and MNEs *(Carvalho et. al, 2018)*

1. Deciding if an IPG has a PNE is Σ_2^p-complete
2. Deciding if an IPG has a MNE is Σ_2^p-complete
3. If $\mathcal{X}^i$ is finite for any player i, there exists an MNE

Knapsack Game *(D. and Scatamacchia, 2023)*

1. Deciding if a Knapsack Game has a PNE is Σ_2^p-complete

An Algorithm

Flexible Modeling

Equilibria Computation

Equilibria Enumeration

Equilibria Selection

Without assuming any specific structure of the game

- Compute PoA/PoS?
- Select (optimize over) a **pure equilibrium**?
- Determine if one exists?

The goal? Zero Regrets

Flexible Modeling

Equilibria Computation

Equilibria Enumeration

Equilibria Selection

Zero Regrets

Given an instance, compute *a* Nash equilibrium minimizing a function $f(x^1, \ldots, x^n)$

Zero Regrets

Given an instance, compute *a* Nash equilibrium minimizing a function $f(x^1, \ldots, x^n)$

Practical assumptions

We can tractably optimize f over $\prod_i \mathcal{X}^i$

We can **linearize** u^i in x^i

High-level idea

1 **Initialization**

$$\mathcal{K} = \{(x,z) : x \in \prod_i \mathcal{X}^i, (x,z) \in \mathcal{L}\} \qquad \Phi := \{0 \leq 1\}$$

2 **Optimization**

$$\bar{x} = \arg \min_{x^1,\dots,x^n,z} \{f(x,z) : (x,z) \in \mathcal{K},\ (x,z) \in \Phi\}$$

3 **Separation**

$$\tilde{x}^i = \arg\min_{x^i} \{u^i(x^i, \bar{x}^{-i}) : x^i \in \mathcal{X}^i\}$$

If there is a player i such that $u^i(\tilde{x}^i, \bar{x}^{-i}) \leq u^i(\bar{x}^i, \bar{x}^{-i})$

$$\Phi = \Phi \cup \{\ u^i(\tilde{x}^i, x^{-i}) \geq u^i(x^i, x^{-i})\ \}$$

2

Else: $\bar{x}$ **is the PNE maximizing** f

Why does it work?

An inequality is an **equilibrium inequality** *if it is valid for the set of Nash equilibria*

$$u^i(\tilde{x}^i, x^{-i}) \geq u^i(x^i, x^{-i}) \quad \forall \tilde{x}^i \in \mathcal{X}^i$$

Theorem *(D. and Scatamacchia, 2022)*

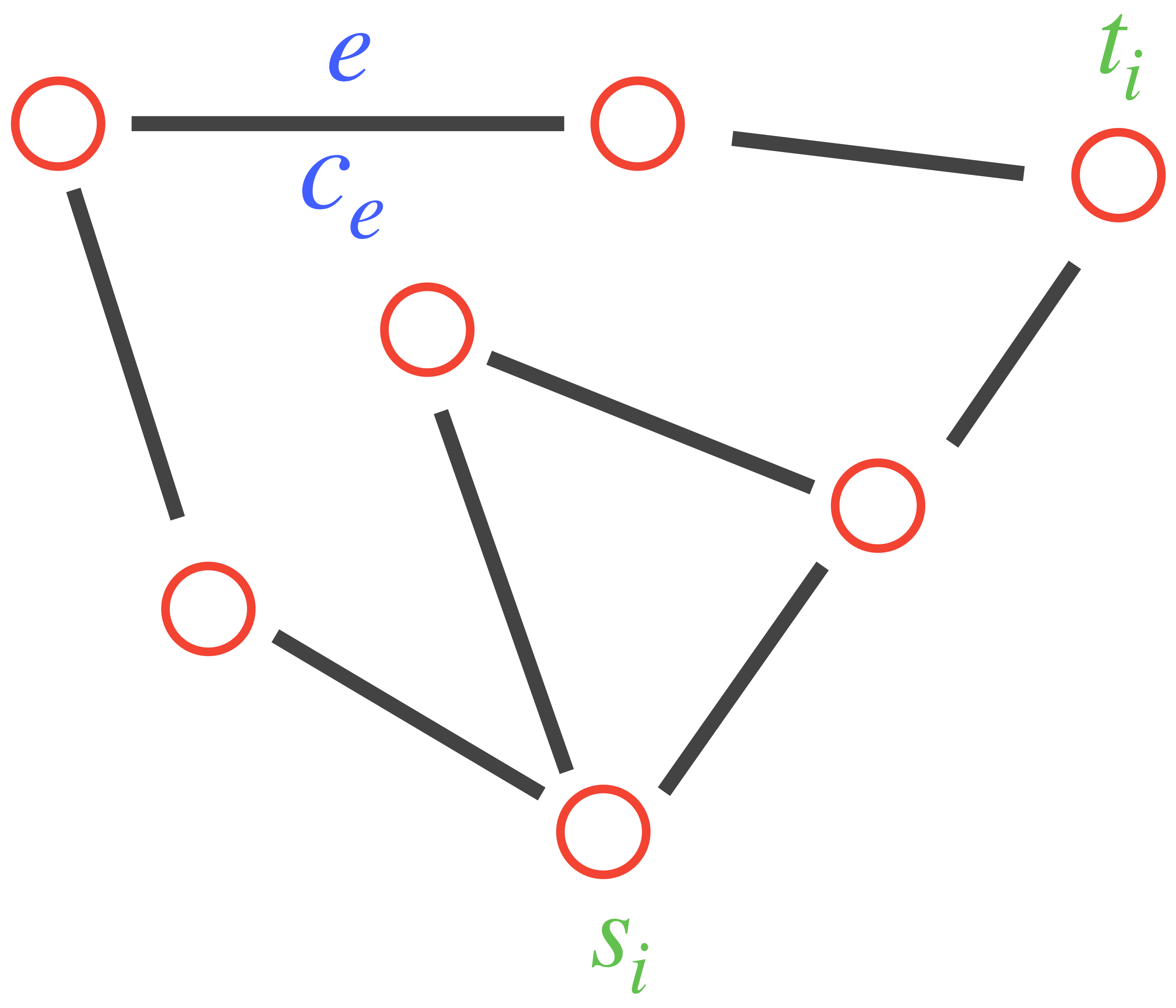
$$P^e := \operatorname{conv}\left(\left\{ (x, z) \in \mathcal{K} : \begin{array}{l} u^i(\tilde{x}^i, x^{-i}) \geq u^i(x^i, x^{-i}) \\ \forall \tilde{x} : \tilde{x}^i \in \mathcal{BR}(i, \tilde{x}^{-i}), \ i = 1, \dots, n \end{array} \right\} \right)$$

(1) P^e is a polyhedron

(2) P^e does not contain feasible "profiles" in its interior

(3) The extreme points of P^e are pure Nash equilibria

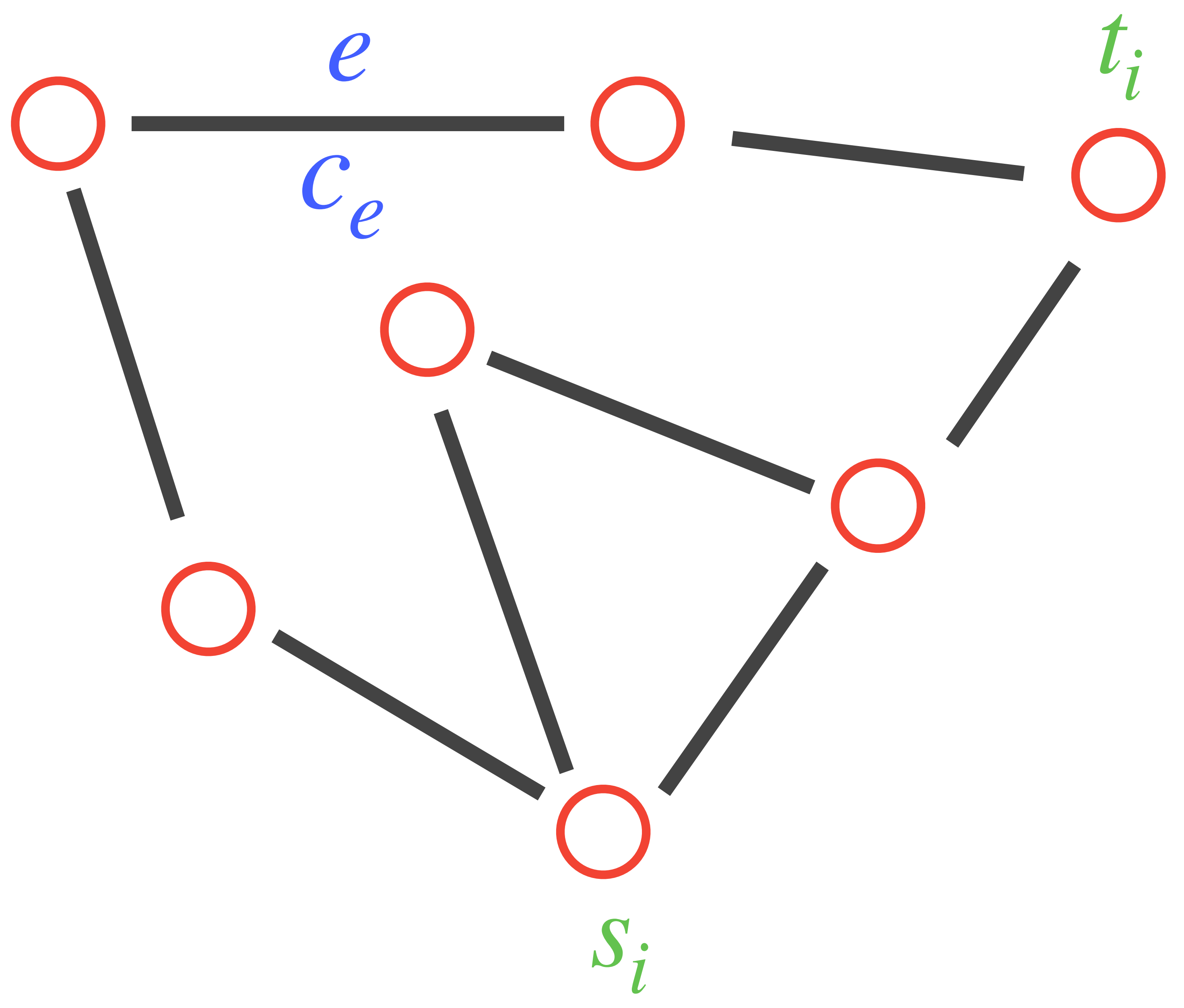
Weighted Network Formation



There are n players optimizing simultaneously the shortest path on a graph $G = (V, E)$ so that:

- The player i needs to go **from s_i to t_i**
- $x_{ie} = 1$ if player i selects the edge $e \in E$
- $\mathcal{X}_i$ are linear flow constraints for the path $s_i \to t_i$
- **The player i has a weight w_i**
- **Players share the cost c_e of building e**

Weighted Network Formation

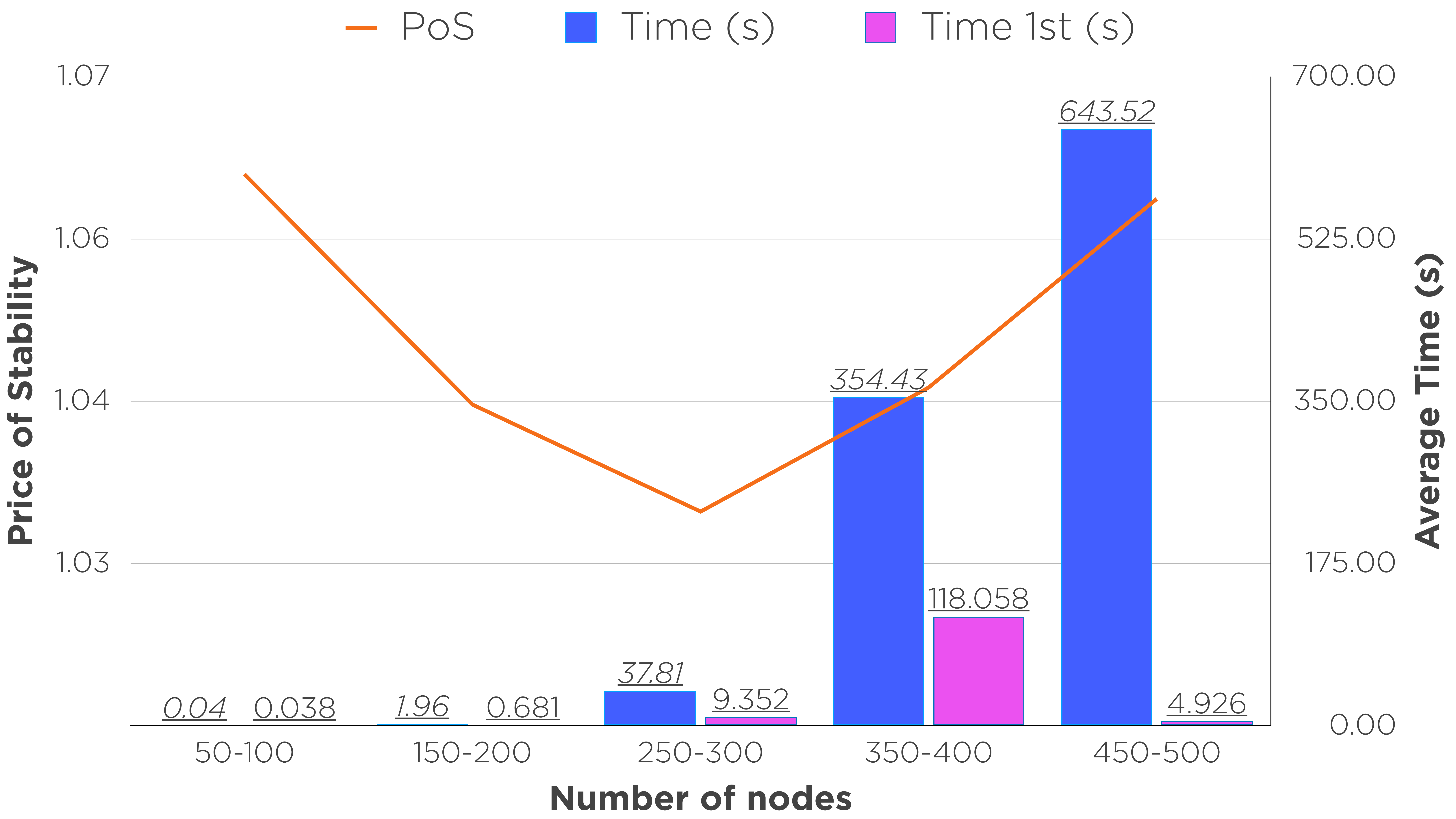


$$\min_{x^i}\{\sum_{e\in E}\frac{w^i c_e x_e^i}{\sum_{k=1}^n w^k x_e^k}:x^i\in\mathcal{X}^i\}.$$

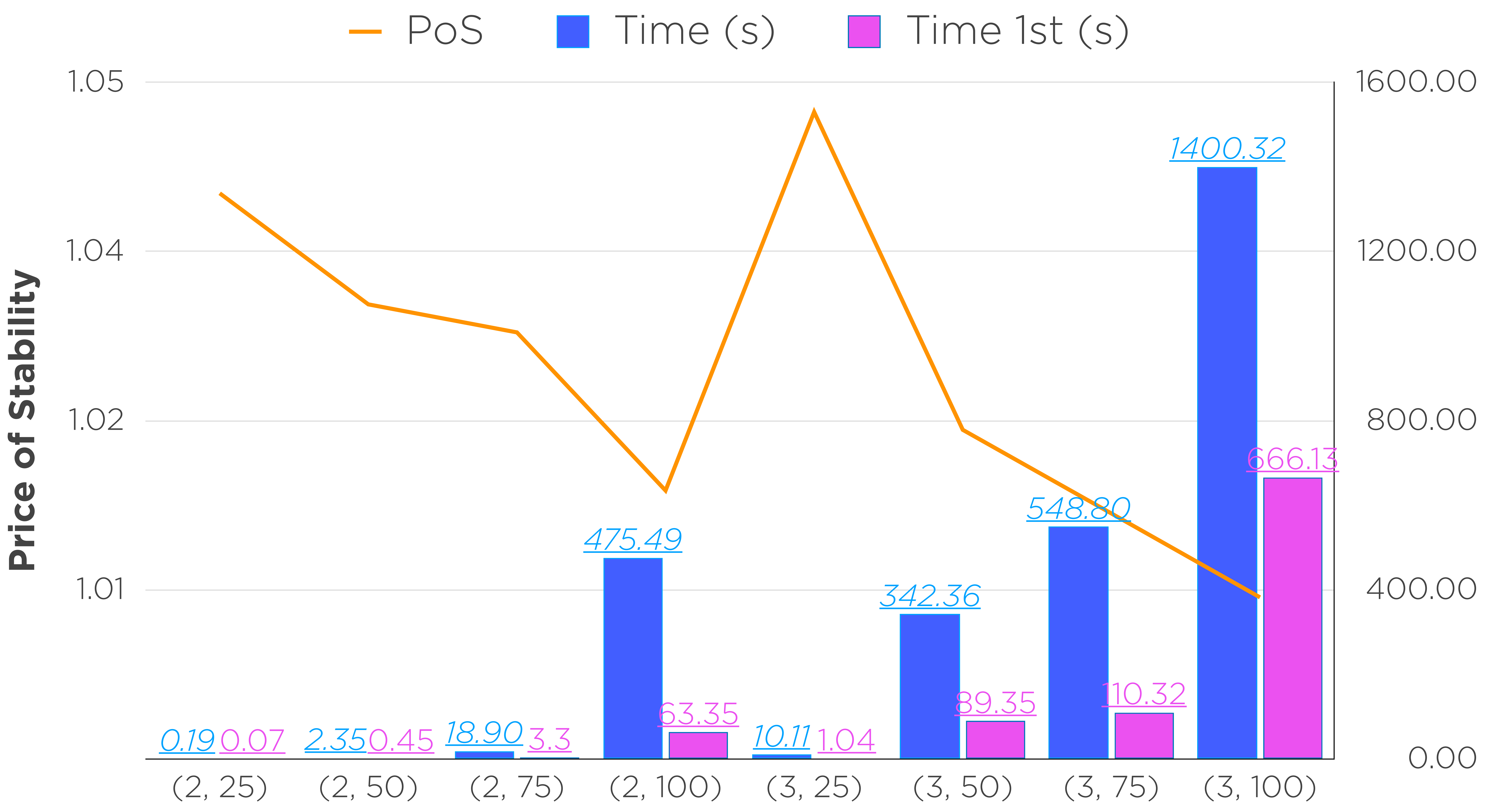
A few remarks

- No algorithms to **select** equilibria in arbitrary NFGs
- Several bounds on PoS/PoA in some specific instances
- We consider the **weighted version** with $n=3$

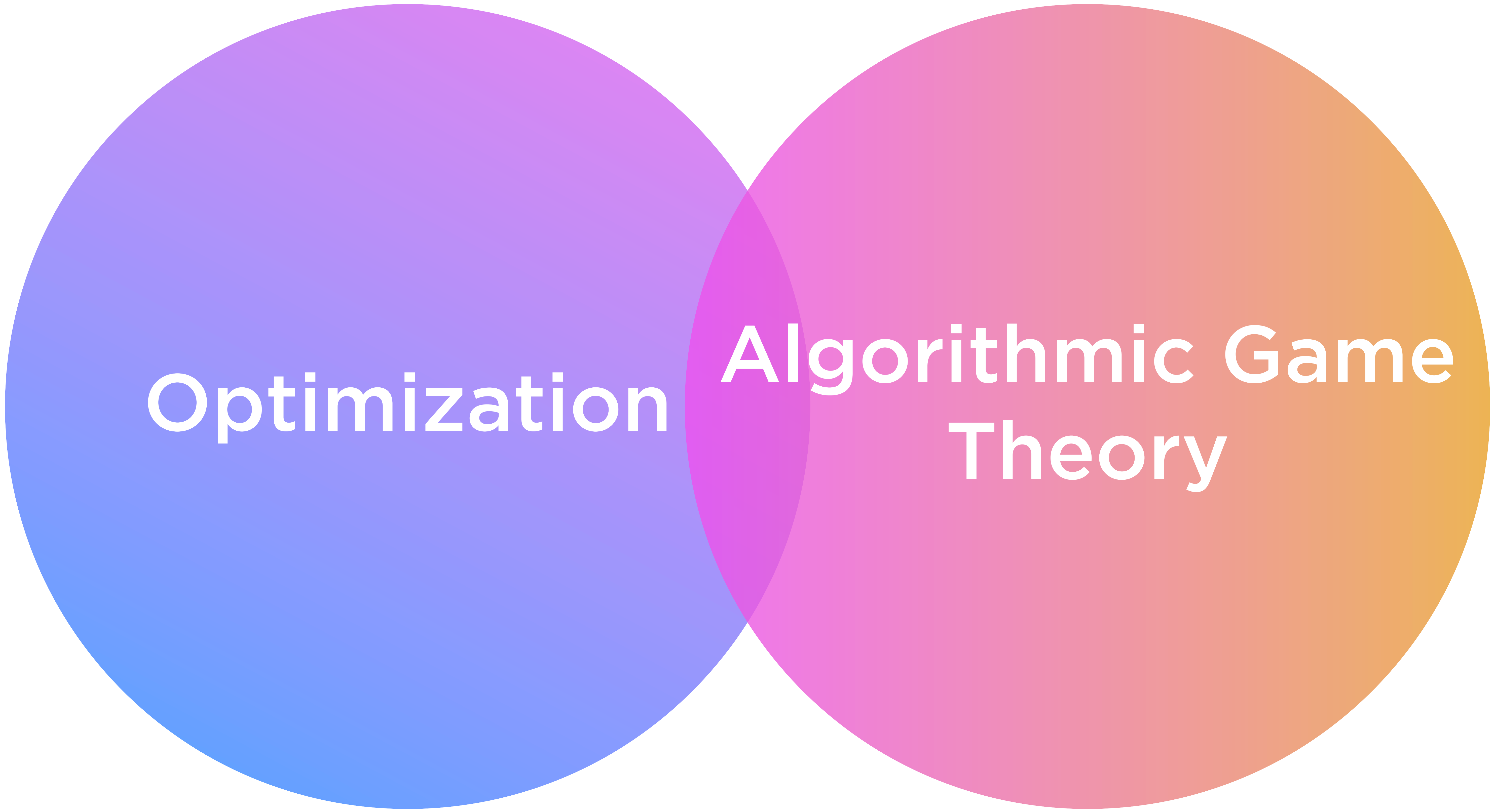
Weighted Network Formation



Knapsack Games



Summing up



Summing up

Algorithmic Game Theory

Model complex and hierarchical structure of **interactions** among agents

Deploy **complex models**, compute their equilibria, and prescribe effective regulatory **interventions**

Optimization

The Zero Regrets Algorithm

INFORMS Journal on Computing - 2023

arXiv 2111.06382

Integer Programming Games: A Gentle Computational Overview

INFORMS 2023 TutORial in O.R. - 2023

arXiv 2303.11188

The Cut-and-Play Algorithm

arXiv 2111.05726

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Knapsack Game (*KPG*)

As for Wizard and Fairy, each player solves a binary Knapsack problem with some **interaction terms** in the objective

$$\max_{x^i} \left\{ \sum_{j=1}^{m} p_j^i x_j^i + \sum_{k=1, k\neq i}^{n} \sum_{j=1}^{m} C_{k,j}^i x_j^i x_j^k : \sum_{j=1}^{m} w_j^i x_j^i \leq b^i, \mathbf{x}^i \in \{0,1\}^m \right\}$$

Knapsack Game (*KPG*)

A few facts:

- No successful attempts to **enumerate or select** equilibria in KPGs with $n > 2$ and $m > 4$ (*Cronert and Minner (2021)*)
- Carvalho et al. (2021, 2022) only compute **an MNE** with at most $n = 3, m \leq 40$
- No results on the complexity of the KPG, nor its PoS/PoA

We select PNEs with $n > 2, m > 50$
We provide "packing" equilibrium inequalities

We prove it is Σ_2^p-complete to determine if a PNE exists + the PoS/PoA are arbitrarily bad

Knapsack Game (*KPG*)

Equilibrium inequalities may also **capture specific structures** or constraint types.

Strategic Payoff Inequalities

A fact

In a packing problem, often the all-zeros strategy is feasible with objective 0

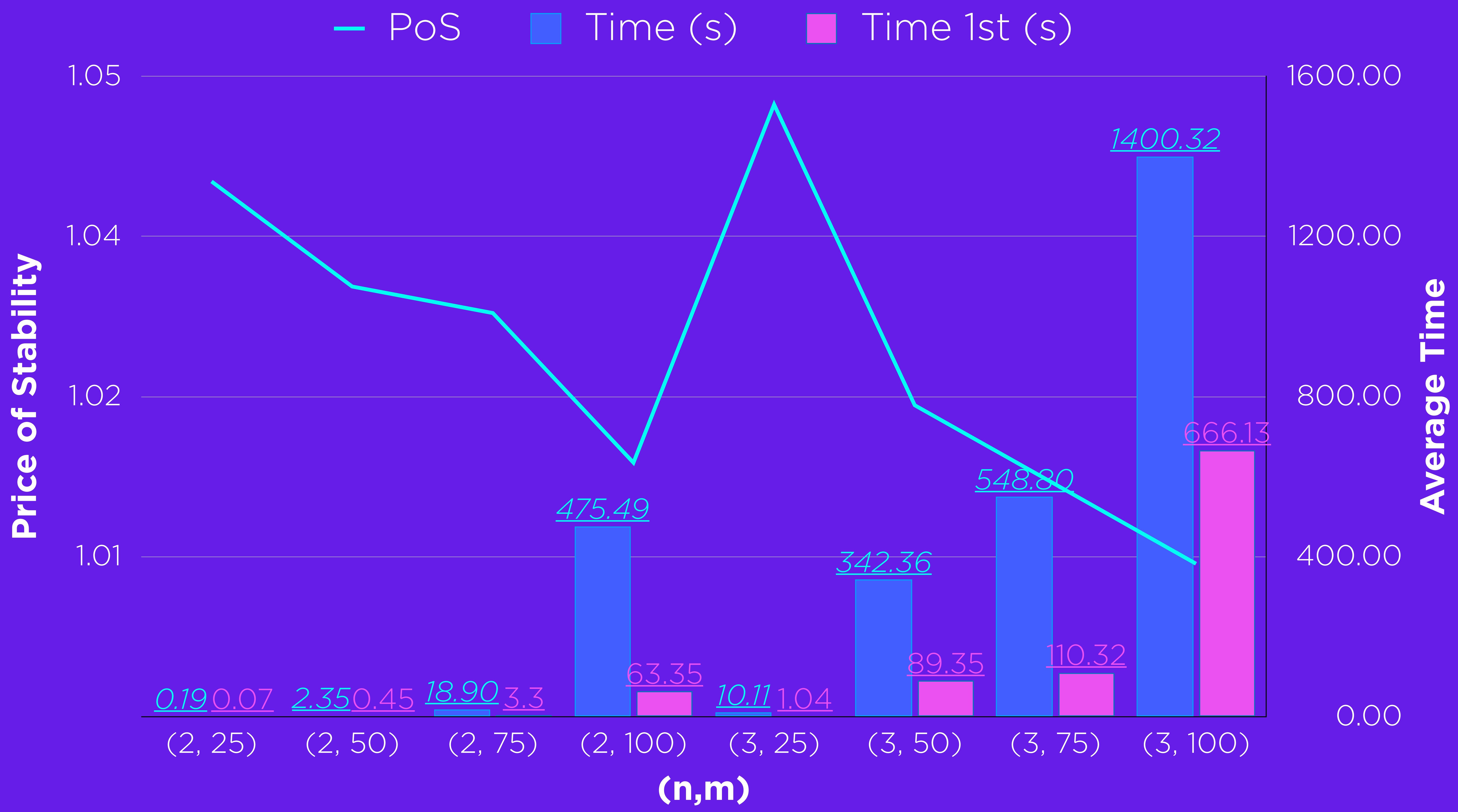
A consequence

Let $\mathcal{S}_i$ be a subset of i's opponents. If $\exists \mathcal{S}_i$ so that

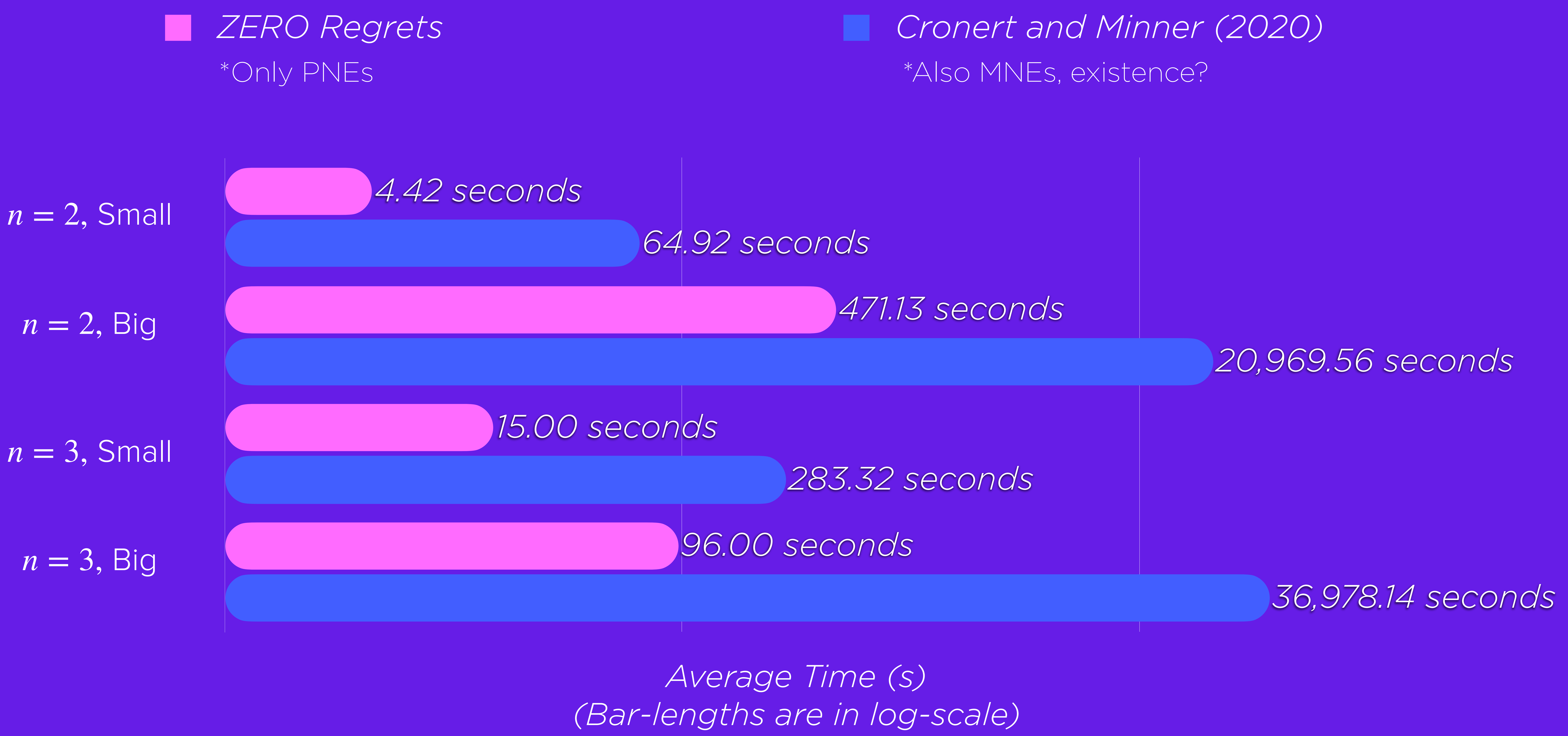
$$p^i_j + \sum_{k \in \mathcal{S}^i_j} C^i_{k,j} < 0,$$

then, $x^i_j + \sum_{k \in \mathcal{S}^i_j} x^k_j \leq |\mathcal{S}^i_j|$ is an **equilibrium inequality**.

Knapsack Game



Facility Location and Design Game



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